Topological Conjugacy

For convenience, let us define a *dynamical endomorphism* to be a piecewise continuous self-map $f : X \to X$ of a complete separable metric space. Sometimes we use the term *endomorphism* for short, although we do not wish to confuse this with other uses of the term (e.g. as in group endomorphism, etc.)

Let $f : X \to X$ and $g : Y \to Y$ be dynamical endomorphisms.

We say that f is *topologically conjugate* to g if there is a homeomorphism h from X onto Y such that $gh = hf$ (or $h^{-1}gh = f$). We sometimes call h a topological conjugacy *between* f and g or *from* f to g. We also say that f and g are topologically conjugate.

Note that topological conjugacy is an equivalence relation on any given collection of dynamical endomorphisms.

We write $f \sim g$ to denote that f is topologically conjugate to g.

A *dynamical property* of a system is one which is preserved under topological conjugacy.

The following are just a few examples of dynamical properties of a given endomorphism f.

1. f has a bounded orbit
2. f has a fixed point
3. f has a dense orbit
4. f has infinitely many periodic orbits
5. the set of periodic orbits of f is dense in the set of bounded orbits

To understand the dynamical properties of a given endomorphism f, one tries to find an understandable *topological model* for f. This is another endomorphism g whose orbit structure is easily describable (or at least many dynamical properties are easily describable) and is such that $g \sim f$.

We now proceed to construct a useful class of topological models for many systems. These are called *Symbolic Systems.*

Let $J_N = \{1, 2, \ldots, N\}$ and let

$$\Sigma_N^+ = J_N^{\mathbf{Z}^+} = \{\mathbf{a} = (a_0 a_1 \ldots) : a_i \in J_N\}.$$

We define a metric d in Σ_N^+ by

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{\infty} \frac{\mid a_i - b_i \mid}{2^i}.$$

It is easy to verify that the pair (Σ_N^+, d) is a compact metric space.

It is actually a Cantor set.

It is elementary to construct a homeomorphism from Σ_2^+ to the middle third set $F(\frac{1}{3})$ we considered earlier.

We define the shift endomorphism $\sigma : \Sigma_N^+ \to \Sigma_N^+$ by

$$\sigma((a_0 a_1 a_2 \ldots)) = (a_1 a_2 \ldots).$$

Then, σ is a continuous onto self-map of Σ_N^+. The pair (σ, Σ_N^+) is called the *full-shift* endomorphism on N symbols, or simply the full one-sided $N-$shift.

Remarks. We can easily describe the periodic points of the full $N-$shift endomorphism σ . Also, it is easy to see that this map has a dense orbit.

Example. Let B_A be the set of bounded orbits for the tent map $f_Z(x) = A(1-2\mid \frac{1}{2} - x \mid)$ for $A > 1$. Then, the pair (f_A, B_A) is topologically conjugate to the full one-sided 2-shift endomorphism (σ, Σ_2^+).

Some special conjugacies

Proposition 0.1 *The logistic map $f(x) = f_4(x) = 4x(1-x)$ is topologically conjugate to the tent map $g(x) = 1 - 2\mid x - \frac{1}{2} \mid$.*

Corollary 0.2 *The logistic map $f_4(x)$ has a dense orbit in $[0,1]$.*

Proof of the Proposition:

Let

$$f_1(x) = 1 - 2\mid x - \tfrac{1}{2} \mid,$$

$$f_2(x) = 2x^2 - 1,$$

$$f_3(x) = 1 - 2x^2,$$

$$f_4(x) = 4x(1-x).$$

It suffices to show that $f_1 \sim f_2$, $f_2 \sim f_3$, and $f_3 \sim f_4$.

Step 1: $f_1 \sim f_2$

Let $T(x) = cos(\pi x)$. Then, T maps $[0,1]$ homeomorphically onto $[-1,1]$. So, Step 1 follows from

$$Tf_1(x) = f_2T(x) \text{ for } x \in [0,1] \tag{1}$$

First, we take $0 < x < \frac{1}{2}$). We have $f_1(x) = 2x$, so

$$\begin{aligned} Tf_1(x) &= cos(2\pi x) \\ &= cos^2(\pi x) - sin^2(\pi x) \\ &= 2cos^2(\pi x) - 1 \\ &= f_2T(x) \end{aligned}$$

Next, consider $\frac{1}{2} < x < 1$. Then, $f_1(x) = 2 - 2x$, so

$$\begin{aligned} Tf_1(x) &= cos(\pi(2-2x)) \\ &= cos(-2\pi x) \\ &= cos(2\pi x) \\ &= 2cos^2(\pi x) - 1 \\ &= f_2T(x) \end{aligned}$$

By continuity, we also have $Tf_1(\frac{1}{2}) = f_2T(\frac{1}{2})$, so (1), and, hence, Step 1 is proved.

Step 2: $f_2 \sim f_3$

Let $T_1(x) = -x$, so that $T_1^{-1} = T_1$.

Then,

$$\begin{aligned} T_1^{-1}f_2T_1(x) &= T_1f_2T_1(x) \\ &= -(2(-x)^2 - 1) \\ &= -(2x^2 - 1) \\ &= 1 - 2x^2 \\ &= f_3(x) \end{aligned}$$

proving Step 2.

Step 3: $f_3 \sim f_4$

Let $T_2(x) = \frac{x+1}{2}$. This is an affine isomorphism taking $[-1, 1]$ onto $[0, 1]$. We will show that $f_4(T_2(x)) = T_2(f_3(x))$, completing the proof of the proposition.

We have

$$\begin{aligned} f_4(T_2(x)) &= 4\frac{x+1}{2}(1 - \frac{x+1}{2}) \\ &= 2(x+1)(\frac{2-(x+1)}{2} \\ &= (x+1)(2-(x+1)) \\ &= 2x + 2 - x^2 - 2x - 1 \\ &= 1 - x^2 \end{aligned}$$

and

$$\begin{aligned} T_2 f_3(x) &= \frac{1 - 2x^2 + 1}{2} \\ &= \frac{2 - 2x^2}{2} \\ &= 1 - x^2. \end{aligned}$$

This completes the proof.

Exercise. Prove that the conjugacy between f_4 and g is unique. That is, if h_1 and h_2 are homeomorphisms such that $h^{-1} f_4 h = g$ and $h_1^{-1} g_4 h_1 = g$, then $h = h_1$.