Linear Fractional Transformations

The theory of Linear Fractional Transformations (LFT’s) in the complex plane is one of the most beautiful and useful tools in complex analysis.

The Schwarzian derivative $Sf$ can be defined for holomorphic maps $f$ (i.e., complex differentiable maps) of the complex plane. It turns out that $Sf \equiv 0$ iff $f$ is a linear fractional transformation.

We recall some of the properties of LFT’s.

**Def.** A linear fractional transformation $T(z)$ of the complex variable $z$ is a map of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d$ are complex numbers such that $ad - bc \neq 0$.

Let $GL(2, \mathbb{C})$ denote the set of $2 \times 2$ complex matrices with non-zero determinant. Note that we can associate a matrix $A \in GL(2, \mathbb{C})$ with $T$ by the assignment

$$T(z) = \frac{az + b}{cz + d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We write this as assignment as the map $T \rightarrow A_T$.

**Exercise.** If $A_T$ and $A_S$ are the matrices associated to the LFT’s $T$ and $S$, respectively, then $A_T \circ S = A_T A_S$. That is, the matrix associated to $T \circ S$ is the product of the matrices associated to $T$ and $S$. Since $GL(2, \mathbb{C})$ is a group under matrix multiplication, it follows that the set of LFT’s is a group under composition.

Also, since the inverse $A^{-1}$ of a $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has the simple formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

we can immediately write the inverse of the LFT

$$T^{-1}(z) = \frac{az + b}{cz + d}.$$
We consider some special LFT’s.
Let \( z \in \mathbb{C} \).
The map \( H(z) \) is

\[
\begin{align*}
\text{a homothety} & \quad \text{if } H(z) = \alpha z \quad \exists \alpha \in \mathbb{R}, \alpha \neq 0 \\
\text{a rotation} & \quad \text{if } H(z) = cz \quad \exists c \in \mathbb{C}, |c| = 1 \\
\text{a translation} & \quad \text{if } H(z) = z + b \quad \exists b \in \mathbb{C} \\
\text{the inversion} & \quad \text{if } H(z) = \frac{1}{z}
\end{align*}
\]

We call these *elementary* LFT’s.

**Exercises.**

1. Every LFT is a composition of elementary ones.

2. A LFT takes lines or circles in \( \mathbb{C} \) onto lines or circles. (Hint: Do this for the elementary ones first. Then use the fact that every LFT is a composition of elementary ones.)

3. A LFT \( T \) is uniquely determined by its image at three distinct points.

Let \( z_1, z_2, z_3 \) be three distinct points in \( \mathbb{C} \). There is a unique LFT \( T \) such that \( T(z_1) = 0, T(z_2) = 1, \) and \( T(z_3) = \infty \).

The formula for \( T \) is

\[
T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}
\]

We identify the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) with the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) using stereographic projection. Here \( S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \).

Let \( x_1 = \Phi(x) \) be the map from \( S^2 \) to \( \mathbb{C} \) obtained as follows. Let \( \ell \) be a line through \((0,0,1)\) which meets the plane \( \{(u_1, u_2, u_3) : u_3 = 0\} \). There are unique points \( x = (x_1, x_2, x_3) \in S^2 \) and \( x_1 \in \mathbb{R}^2 \) which lie on \( \ell \). Set \( \Phi(x) = x_1 \).

Identifying the point \((x, y) \in \mathbb{R}^2 \) with \( x + iy \in \mathbb{C} \), there is a simple formula for the map \( \Phi \).

If we let \( x = (x_1, x_2, x_3) \in S^2 \), (so \( x_1^2 + x_2^2 + x_3^2 = 1 \),) then

\[
\Phi(x) = \frac{x_1 + ix_2}{1 - x_3}.
\]
We leave the verification of this as an exercise (see Ahlfors, Complex Analysis).

Observe that

- \( x \) in upper hemisphere \( \Rightarrow |\Phi(x)| > 1 \)
- \( x \) in lower hemisphere \( \Rightarrow |\Phi(x)| < 1 \)
- \( x \) in \( S^2 \cap \mathbb{R}^2 \) \( \Rightarrow \Phi(x) = x \)

We think of the point \((0, 0, 1)\) in \( S^2 \) as the geometric representation of the point at infinity in the extended complex plane \( \mathbb{C} \).

The map \( \Phi \) (or its inverse) is called stereographic projection. It is obvious that \( \Phi^{-1} \) carries lines in \( \mathbb{C} \) to circles through \((0, 0, 1)\). It can also be shown that \( \Phi^{-1} \) takes circles in \( \mathbb{C} \) onto circles on \( S^2 \).

Let \( T \) be a LFT. Then, \( T \) takes lines or circles in \( \mathbb{C} \) into other lines or circles. So, the lifted map to \( S^2 \) defined by \( \Phi^{-1}T\Phi \) takes circles to circles on \( S^2 \).

**Exercises.**

1. Find linear fractional transformations carrying the sets \( E_1 \) onto \( E_2 \) where

   (a) \( E_1 = \{ z : |z| < 1 \}, E_2 = \{ z : \text{Im}(z) > 0 \} \).

   (b) \( E_1 = \{ z : |z-i| < \frac{1}{2} \}, E_2 = \{ z : |z-1| < 4 \} \).

2. Consider the subset \( \mathcal{U} \) of LFT’s \( T_a \) of the form

   \[
   T_a(z) = \frac{z - a}{1 - \overline{a}z}
   \]

   where \( a \) is a complex number of norm less than 1, and \( \overline{a} \) is its complex conjugate.

   (a) Show that \( \mathcal{U} \) is a subgroup of the group of all LFT’s.

   (b) Show that each \( T_a \in \mathcal{U} \) takes the unit circle \( \{ z : |z| = 1 \} \) onto itself.