1 Systems of Differential Equations

Let U be an open subset of $\mathbf{R}^n$, I be an open interval in $\mathbf{R}$ and : $\mathbf{I}\times\mathbf{R^n} \to \mathbf{R^n}$ be a function from $I \times \mathbf{R}^n$ to $\mathbf{R}^n$.

The equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

is called a first order ordinary differential equation in $\mathbf{R}^n$. We emphasize here that x is an $n-$dimensional vector in R^n. We also consider the *initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}),\ x(t_0) = x_0 \tag{2}$$

where $t_0 \in I$ and $\mathbf{x}_0 \in U$.

A *solution* to the IVP (2) is a differentiable function $\mathbf{x}(t)$ from an open subinterval $J \subset I$ containing t_0 such that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$$

for $t \in J$.

The *general solution* to (1) is an expression

$$\mathbf{x}(t, \mathbf{c}) \tag{3}$$

where $\mathbf{c}$ is an $n-$dimensional constant vector in $\mathbf{R}^n$ such that every solution of (1) can be written in the form (3) for some choice of $\mathbf{c}$.

If we write out the D.E. (1) in coordinates, we get a system of first order differential equations as follows.

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{4}$$

Fact: The $n-$th order scalar D.E. is equivalent to a simple $n-$dimensional system.

Consider

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}). \tag{5}$$

Letting $x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}$, we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(t, x_1, \ldots, x_n) \end{aligned} \tag{6}$$

If we have a solution $y(t)$ to (5), and set $x_1 = y(t), x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t)$, then $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ is a solution to the system (6). Conversely, if we have a solution $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ to the system 6, then putting $y(t) = x_1(t)$ gives a solution to (5).

The following existence and uniqueness theorem is proved in more advanced courses.

Theorem(Existence-Uniqueness Theorem for systems). *U be an open set in $\mathbf{R}^n$, and let I be an open interval in $\mathbf{R}$. Let $\mathbf{f}(t, \mathbf{x})$ be a C^1 function of the variables $(t, \mathbf{x})$ defined in $I \times U$ with values in $\mathbf{R}^n$. Let $(t_0, \mathbf{x}_0) \in I_0 \times U$. Then, there is a unique solution $x(t)$ to the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}_0.$$

If the right side of the system $\mathbf{f}(t, \mathbf{x})$ does not depend on time, then one calls the system *autonomous* (or time-independent). Otherwise, one calls the system *non-autonomous* or time-dependent.

There is a simple geometric description of autonomous systems in $\mathbf{R}^n$. In that case, we consider

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7}$$

where $\mathbf{f}$ is a C^1 function defined in an open subset U in $\mathbf{R}^n$. We think of $\mathbf{f}$ as a *vector field* in U and solutions $\mathbf{x}(t)$ of (7) as curves in U which are everywhere tangent to $\mathbf{f}$.

1.1 Linear Systems of Differential Equations: General Properties

The system

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{g}(t) \tag{8}$$

in which $A(t)$ is a continuous $n \times n$ matrix valued function of t and $\mathbf{g}(t)$ is a continuous $n-$vector valued function of t is called a linear system of differential equations (or a linear differential equation) in $\mathbf{R}^n$.

As in the case of scalar equations, one gets the general solution to (8) in two steps. First, one finds the general solution $\mathbf{x}_h(t)$ to the associated homogeneous system

$$\dot{\mathbf{x}} = A(t)\mathbf{x}. \tag{9}$$

Then, one finds a particular solution $\mathbf{x}_p(t)$ to (8) and gets the general solution to (8) as a sum

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t).$$

Accordingly, we will examine ways of doing both tasks.

Let $\mathbf{y}_i(t)$ be a collection of $\mathbf{R}^n-$valued functions for $1 \leq i \leq k$. We say that they form a *linearly independent* set of functions if whenever $c_1, \ldots, c_k$ are k scalars such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \ldots + c_k\mathbf{y}_k(t) = \mathbf{0}$$

for all t, we have that $c_1 = \ldots = c_k = 0$.

An $n \times n$ matrix $\Phi(t)$ of linearly independent solutions to the homogeneous linear system (9) is called a *fundamental matrix* for (9).

A necessary and sufficient condition for the matrix of solutions $\Phi(t)$ to be a fundamental matrix is that $det(\Phi(t)) \neq 0$ for some (or any) t.

If $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ are n solutions to (9), and $\Phi(t)$ is the matrix whose columns are the functions $\mathbf{y}_i(t)$, then the function

$$W(t) = W(\mathbf{y}_1, \ldots, \mathbf{y}_n)(t) = det(\Phi(t))$$

is called the *Wronskian* of the collection $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)\}$ of solutions. It is then a fact that $W(t)$ vanishes at some point t_0 if and only if it vanishes at all point t.

The general solution to (9) has the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}$$

where $\Phi(t)$ is *any* fundamental matrix for (9) and $\mathbf{c}$ is a constant vector.

Thus, we have to find fundamental matrices and particular solutions. We will do this explicitly below for $n = 2, 3$ and constant matrices A.

To close this section, we observe an analogy between systems

$$\mathbf{x}' = A\mathbf{x}$$

and the scalar equation $x' = ax$.

One can define the matrix $exp(A) = e^A$ by the power series

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots.$$

It can be shown that the matrix series on the right side of this equation converges for any A. The series represents a matrix with many properties analogous to the usual exponential function of a real variable.

In particular, for a real number t, the matrix function $t \to e^{tA}$ is a differentiable matrix valued funtion and its derivative, computed by differentiating the series

$$I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \dots$$

term by term satisfies

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

It follows that, for each vector $\mathbf{x}_0$, the vector function

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0$$

is the unique solution to the IVP

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0.$$

Hence, the matrix e^{tA} is a fundamental matrix for the system

$$\mathbf{x}' = A\mathbf{x}.$$

This observation is useful in certain circumstances, but, in general, it is hard to compute e^{tA} directly. In practice the methods involving eigenvalues described in the next secion are easier to use to find the general solution to

$$\mathbf{x}' = A\mathbf{x}.$$

Linear Homogeneous Systems with Constant Coefficients

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{10}$$

where A is a constant $n \times n$ matrix and $\mathbf{x}$ is an $n-$vector in $\mathbf{R}^n$.

In the one-dimensional (scalar case), we found solutions using exponential functions, so it seems reasonable to try to find a solution of the form

$$\mathbf{x}(t) = e^{rt}\xi$$

where r is a real constant and ξ is a non-zero constant vector.

Plugging in, we get

$$\dot{\mathbf{x}}(t) = re^{rt}\xi = Ae^{rt}\xi$$

for all t. Since e^{rt} is never zero, we can cancel it and get

$$r\xi = A\xi$$

or

$$(rI - A)\xi = \mathbf{0}. \tag{11}$$

Thus, r is a scalar such that there is a non-zero vector ξ such that ξ is a solution of the system of linear equations (11).

Thus, r must be an eigenvalue of A with associated eigenvector ξ (see section 16 for the definitions and properties).

Let us first consider the case in which A has a real eigenvalue with associated eigenvector ξ.

Then, reversing the above steps shows that the function $\mathbf{x}(t) = e^{rt}\xi$ is a solution to (10).

Now, suppose that the characteristic polynomial $z(r) = det(rI - A)$ has n distinct real roots.

That is, we suppose that

$$z(r) = (r - r_1)(r - r_2)\cdots(r - r_n)$$

where $r_i \neq r_j$ for $i \neq j$.

Then, letting $\mathbf{v}_i$ be an eigenvector for r_i for each i, we get that each function $\mathbf{x}_i(t) = e^{r_i t}\mathbf{v}_i$ is a solution, and these are linearly independent functions since their Wronskian $W(t)$ at $t = 0$ equals $det(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ which is the determinant of the matrix whose columns are the $\mathbf{v}_i's$. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent since these are eigenvectors for distinct eigenvectors.

Thus, the general solution to (10) has the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

In the cases of multiple roots or complex roots, one has to do some extra work and we will consider those situations later in the high dimensional case.

For now, let us specialize to the case of two dimensional systems.

2 Two dimensional homogeneous systems of linear differential equations with constant coefficients

Consider the system

$$\begin{aligned} \dot{x} &= a\,x + b\,y \\ \dot{y} &= c\,x + d\,y \end{aligned}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a constant 2×2 real matrix.

We compute the eigenvalues r_1, r_2. These are the roots of the characteristic polynomial

$$r^2 - tr(A)r + det(A).$$

Case 1: Both roots are real and distinct. Say these are $r_1 > r_2$.

Step 1. Compute the eigenvectors $\mathbf{v}_1$ for r_1 and $\mathbf{v}_2$ for r_2, repectively.

Then, we get solutions of the form

$$\mathbf{x}_1 = e^{r_1 t}\mathbf{v}_1, \ \mathbf{x}_2 = e^{r_2 t}\mathbf{v}_2.$$

These turn out to be linearly independent, so the general solution is

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{v}_1 + c_2 e^{r_2 t}\mathbf{v}_2$$

where c_1 and c_2 are constants.

In this case, just as we did for second order scalar equations, we call $\mathbf{x}_1(t)$ the *first fundamental solution* to (2) and $\mathbf{x}_2(t)$ the *second fundamental solution* to (2).

Case 2: Both roots are real and equal. Say the common root is r_1.

We get one solution $\mathbf{x}_1(t)$ of the form

$$\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1$$

where $\mathbf{v}_1$ is an eigenvector for r_1.

Next, we have two subcases

Subcase 2a: There are two linearly independent eigenvectors, say ξ, η for the eigenvalue r_1.

In this case the general solution is

$$\mathbf{x}(t) = e^{r_1 t}(c_1\xi + c_2\eta).$$

An example of this is the system

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= 2y \end{aligned}$$

with general solution

$$\mathbf{x}(t) = e^{2t}\left(c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Subcase 2b: All eigenvectors for r_1 are multiples of $\mathbf{v}_1$.

In this case we proceed as follows.

Let us try to find another linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{r_1 t}\mathbf{v}_0 + te^{r_1 t}\mathbf{v}_1.$$

We get

$$\begin{aligned}\dot{\mathbf{x}}_2(t) &= r_1 e^{r_1 t}\mathbf{v}_0 + tr_1 e^{r_1 t}\mathbf{v}_1 + e^{r_1 t}\mathbf{v}_1 \\ &= A(e^{r_1 t}\mathbf{v}_0 + te^{r_1 t}\mathbf{v}_1),\end{aligned}$$

or

$$r_1\mathbf{v}_0 + tr_1\mathbf{v}_1 + \mathbf{v}_1 = A(\mathbf{v}_0 + t\mathbf{v}_1).$$

Setting the constant and terms with t equal we get that $\mathbf{v}_1$ is an eigenvector, and $\mathbf{v}_0$ satisfies the linear system

$$(A - r_1 I)\mathbf{v}_0 = \mathbf{v}_1. \tag{12}$$

Finding the solution v_0, we can in fact obtain a second linearly independent solution of the above form.

Thus, the general solution has the form

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{v}_1 + c_2 e^{r_1 t}(\mathbf{v}_0 + t\mathbf{v}_1).$$

where $\mathbf{v}_1$ is an eigenvector associated to r_1 and $\mathbf{v}_0$ satisfies (12).

Note that this involves solving the two systems of equations

$$(A - r_1 I)\mathbf{v}_1 = \mathbf{0},\ (\mathbf{A} - \mathbf{r_1 I})\mathbf{v_0} = \mathbf{v_1}.$$

Example.

Consider the system

$$\begin{aligned}\dot{x} &= 4x - y \\ \dot{y} &= x + 6y\end{aligned}$$

The matrix is

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}$$

with characteristic equation

$$r^2 - 10r + 25 = (r-5)^2.$$

Hence, $r = 5$ is a root of multiplicity two.

Since the upper right corner of the matrix is not zero, we can obtain an eigenvector $\mathbf{v}_1$ of the form

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \frac{r-a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5-4}{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus, we get one non-zero solution as

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For the second independent solution we have

$$\mathbf{x}_1(t) = e^t \mathbf{v}_0 + te^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where

$$(A - 5I)\mathbf{v}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{13}$$

or,

$$\begin{pmatrix} 4-5 & -1 \\ 1 & 6-5 \end{pmatrix} \mathbf{v}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{14}$$

In scalar form, with $\mathbf{v}_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, this is

$$\begin{aligned} -v_1 - v_2 &= 1 \\ v_1 + v_2 &= -1 \end{aligned}$$

Note that these two equations are multiples of each other (that will always be the case with a root of multiplicity two and $b \neq 0$).

So, we can use the first equation. Setting $v_1 = 1$, we get $v_2 = -2$.

The general solution is

$$\mathbf{x}(t) = c_1 e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

The solutions to this last equation are all vectors of the form $\begin{pmatrix} 1 \\ \xi_2 \end{pmatrix}$ with ξ_2 arbitrary, so we can pick $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and get a second linearly independent solution as

$$\mathbf{x}_2(t) = e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Note that we could also choose $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and get a second linearly independent solution as

$$\mathbf{x}_2(t) = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + te^t\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This last choice is consistent with the method using *first* and *second fundamental* solutions we will define later.

Remark. Note that we used the method above when there are not two linearly independent eigenvectors for the eigenvalue 1. We did not check whether this is the case, so why does this work? The answer is that if there were indeed two linearly independent eigenvectors for the eigenvalue 1, then the system (13) would not have had any solutions, so the fact that we could solve the system justifies the approach. (The proof of this requires more linear algebra and will have to be deferred to a more advanced course.)

This method generalizes to n dimensional systems with eigenvalues of multiplicity greater than one although the linear algebra required is more complicated.

Case 3: The roots are $\alpha \pm i\beta$ where $\beta > 0$.

Here we use complex variables. We have a complex solution of the form

$$\mathbf{x}_{1c}(t) = e^{(\alpha+i\beta)t}\xi$$

where ξ is the complex eigenvector

$$\begin{bmatrix} 1 \\ \frac{\alpha+i\beta-a}{b} \end{bmatrix}$$

associated to the eigenvalue $\alpha+i\beta$. (Note that here we used the condition $b \neq 0$. It is possible to show that, for 2×2 real matrices with non-real complex conjugate eigenvalues, the off diagonal terms b and c are, in fact, non-zero).

To find two independent solutions to (10), in this case, one need only take the real and imaginary parts of the complex solution $\mathbf{x}_c(t)$.

Thus, if $\mathbf{x}_1(t)$ is this real part and $\mathbf{x}_2(t)$ is this complex part, then the general solution will have the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

If we write express the complex eigenvector ξ as $\mathbf{u} + i\mathbf{v}$, then

$$\mathbf{x}_1(t) = e^{\alpha t}[cos(\beta t)\mathbf{u} - sin(\beta t)\mathbf{v}]$$

and

$$\mathbf{x}_2(t) = e^{\alpha t}[cos(\beta t)\mathbf{v} + sin(\beta t)\mathbf{u}]$$

Let us do an example.

Consider the system

$$\begin{aligned} \dot{x} &= 2\,x - 3\,y \\ \dot{y} &= 2\,x + 4\,y \end{aligned}$$

The matrix A is given by

$$\begin{pmatrix} 2 & -3 \\ 2 & 4 \end{pmatrix}.$$

The characterstic polynomial is

$$r^2 - 6r + 14$$

with roots

$$r = \frac{6 \pm \sqrt{36 - 56}}{2} = 3 \pm i\sqrt{5}.$$

We seek a complex eigenvector $\xi = (\xi_1, \xi_2)$ for the eigenvalue $r = 3 + i\sqrt{5}$. We get the equation

$$(rI - A)\xi = \mathbf{0}.$$

or

$$\begin{pmatrix} r-2 & 3 \\ -2 & r-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because the matrix is singular, we need only consider the first row equation

$$(r-2)\xi_1 + 3\xi_2 = 0$$

Setting $\xi_1 = 1$, we get

$$\begin{aligned} \xi_2 &= \frac{(2-r)}{3} \\ &= \frac{2 - (3 + i\sqrt{5})}{3} \\ &= \frac{-1 - i\sqrt{5}}{3} \end{aligned}$$

Thus, we get the first fundamental solution of the form

$$\begin{aligned} \mathbf{x}_1 &= e^{(3+i\sqrt{5})t} \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{5}}{3} \end{pmatrix} \\ &= e^{(3+i\sqrt{5})t} \left(\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} + i \begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \right) \end{aligned}$$

and the second fundamental solution of the form

$$\begin{aligned}\mathbf{x}_2 &= e^{(3-i\sqrt{5})t}\begin{pmatrix} 1 \\ \frac{-1+i\sqrt{5}}{3} \end{pmatrix} \\ &= e^{(3+i\sqrt{5})t}\left(\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} - i\begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix}\right)\end{aligned}$$

The real and imaginary parts of $x_1(t)$ are

$$R_1 \stackrel{\text{def}}{=} e^{3t}\left(\cos(\sqrt{5}t)\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} - sin(\sqrt{5}t)\begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix}\right)$$

$$I_1 \stackrel{\text{def}}{=} e^{3t}\left(\sin(\sqrt{5}t)\begin{pmatrix} 1 \\ \frac{-1}{3} \end{pmatrix} + \cos(\sqrt{5}t)\begin{pmatrix} 0 \\ -\frac{\sqrt{5}}{3} \end{pmatrix}\right)$$

The general real solution is

$$\mathbf{x}(t) = c_1 R_1 + c_2 I_1.$$

3 The first and second fundamental solutions and the complex solution

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}$$

with characteristic polynomial $z(r) = r^2 - (a+d)r + ad - bc$.

For convenience in finding the general solution to this system, we define the *first* and *second* fundamental solutions.

The general solution can be expressed as a linear combination of these two solutions.

The definitions depend on the nature of the roots. We assume that $b \neq 0$. Corresponding solutions can be defined if $b = 0$ and $c \neq 0$, but, for the sake of simplicity, we avoid that here. If fact, one could simply change coordinates, interchanging x and y and reduce to the present case.

Case 1: real distinct roots $r_1 > r_2$, $b \neq 0$

First Fundamental Solution: $\mathbf{x}_1(t) = e^{r_1 t}\begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix}$

Second Fundamental Solution: $\mathbf{x}_2(t) = e^{r_2 t}\begin{pmatrix} 1 \\ \frac{r_2 - a}{b} \end{pmatrix}$

Case 2: real equal roots $r_1 = r_2$, $b \neq 0$

First Fundamental Solution: $\mathbf{x}_1(t) = e^{r_1 t}\begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix}$

Second Fundamental Solution: $\mathbf{x}_2(t) = te^{r_1 t}\begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix} + e^{r_1 t}\begin{pmatrix} 1 \\ \frac{r_1 - a + 1}{b} \end{pmatrix}$

Case 3: complex roots $r = \alpha + \beta i$ with $\beta > 0$.

Complex Solution: $\mathbf{x}_c(t) = e^{t(\alpha+\beta i)}\begin{pmatrix} 1 \\ \frac{\alpha+i\beta-a}{b} \end{pmatrix} = e^{t(\alpha+\beta i)}\left[\begin{pmatrix} 1 \\ \frac{\alpha-a}{b} \end{pmatrix} + i\begin{pmatrix} 0 \\ \frac{\beta}{b} \end{pmatrix}\right]$

First Fundamental Solution: $\mathbf{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\begin{pmatrix} 1 \\ \frac{\alpha-a}{b} \end{pmatrix} - \sin(\beta t)\begin{pmatrix} 0 \\ \frac{\beta}{b} \end{pmatrix}\right]$

Second Fundamental Solution: $\mathbf{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\begin{pmatrix} 1 \\ \frac{\alpha-a}{b} \end{pmatrix} + \cos(\beta t)\begin{pmatrix} 0 \\ \frac{\beta}{b} \end{pmatrix}\right]$

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4 An alternate method for 2 dimensional systems: Elimination and reduction to scalar equations

First, we give some examples to describe the elimination method to solve two dimensional systems.

Example 1:

Consider the system

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= x - y \end{aligned}$$

We can write

$$y = \dot{x} - x \tag{15}$$

from the first equation and substitute into the second equation getting

$$\begin{aligned} \dot{y} &= \ddot{x} - \dot{x} \\ &= x - (\dot{x} - x) \end{aligned}$$

This gives the second order scalar equation for x

$$\ddot{x} - 2x = 0.$$

We know how to solve this. The characteristic equation is $r^2 - 2$ **with roots** $r = \pm\sqrt{2}$ **and general solution**

$$x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

Then, we get y **from (15) as**

$$\begin{aligned} y(t) &= \dot{x} - x \\ &= c_1\sqrt{2}e^{\sqrt{2}t} - c_2\sqrt{2}e^{-\sqrt{2}t} - c_1 e^{\sqrt{2}t} - c_2 e^{-\sqrt{2}t}, \end{aligned}$$

so, the general solution to the system is

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} \\ c_1\sqrt{2}e^{\sqrt{2}t} - c_2\sqrt{2}e^{-\sqrt{2}t} - c_1 e^{\sqrt{2}t} - c_2 e^{-\sqrt{2}t} \end{pmatrix} \\ &= c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}.\end{aligned}$$

This method can most often be used for two dimensional homogeneous systems.

Which method is best?

In my opinion, this method is best when the eigenvalue is real of multiplicity two and the matrix method is best in the other cases.

Example.

Let us apply the method of elimination to the system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= x + y\end{aligned}$$

we considered above.

We get

$$x = \dot{y} - y$$

$$\dot{x} = \ddot{y} - \dot{y} = x = \dot{y} - y.$$

or

$$\ddot{y} - 2\dot{y} + y = 0.$$

The general solution is

$$y(t) = c_1 e^t + c_2 t e^t.$$

This gives

$$\begin{aligned}x(t) &= \dot{y} - y \\ &= c_1 e^t + c_2 e^t + c_2 t e^t - (c_1 e^t + c_2 t e^t) \\ &= c_2 e^t\end{aligned}$$

and the general solution

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} c_2 e^t \\ c_1 e^t + c_2 t e^t \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ t \end{pmatrix}.\end{aligned}$$

Finally, we describe some general aspects of the method of elimination.

We consider the system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

We assume, that either b or c is not 0. Otherwise, the system is diagonal and easily solvable.

Assuming $b \neq 0$, we use the elimination method to find a second order equation for $x(t)$. We will see that the characteristic polynomial for this second order equation is the same as the characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The latter characteristic polynomial is

$$r^2 - (a+d)r + ad - bc.$$

Now,

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + bcx + bdy \\ &= ax' + dx'((bc - ad)x \end{aligned}$$

or

$$x'' - (a+d)x' + ad - bc = 0.$$

Now, we can find the general solution $x(t)$ of this second order equation, and then get $y(t)$ in the original system from

$$y = \frac{x' - ax}{b}$$

If, $b = 0$ but $c \neq 0$, we use the elimination method to find a second order equation for $y(t)$. We get the general solution for $y(t)$, and then get $x(t)$ from

$$x = \frac{y' - dy}{c}.$$