12d. Regular Singular Points

We have studied solutions to the linear second order differential equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the cases with P, Q, R real analytic in a neighborhood of a point $x = a$ and $P(a) \neq 0$.

In that case, we found that we could expand the solutions as power series of the form

$$y(x) = \sum_{n \geq 0} c_n x^n$$

and, with the knowlegde of c_0 and c_1, we could determine, successively the coefficients c_n with $n \geq 2$ by certain recurrence relations.

There many important linear second order differential equations (with non-constant coefficients) which arise in mathematical physics which do not satisfy the above conditions, and we would like to be able to obtain power series solutions for some of these.

The method we study here was discovered by the German mathematician F. G. Frobenius in the 1870's and is therefore often called the *Frobenius method*.

Recall that $x = a$ is called a singular point of (1) if $P(a) = 0$.

This causes difficulties in writing (1) in the standard form

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \tag{2}$$

The assumptions that Frobenius found to enable one to get around this difficulty are, roughly speaking, that the denominators of $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are not too bad.

Let us be more precise.

We say that the singular point $x = a$ is a *regular singular point* of equation (1) if the equation can be written as

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \tag{3}$$

where

$$\lim_{x\to a}\frac{(x-a)Q(x)}{P(x)} \text{ and } \lim_{x\to a}\frac{(x-a)^2R(x)}{P(x)}$$

both exist and are *finite*.

This means that, for x near a, the functions $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ can be written as

$$\frac{Q(x)}{P(x)} = \frac{Q_1(x)}{(x-a)}$$

and

$$\frac{R(x)}{P(x)} = \frac{R_1(x)}{(x-a)^2}$$

where $Q_1(x)$ and $R_1(x)$ are analytic at $x = a$.

This is the same as saying that

> the multiplier of y' in (3) has at most the factor $(x-a)$ in its denominator

and

> the multiplier of y in (3) has at most the factor $(x-a)^2$ in its denominator

A singular point which is not regular will be called *irregular*.

We will make use of the following fact: If $P(x)$ and $Q(x)$ are analytic at a and $P(a) \neq 0$, then $\frac{Q(x)}{P(x)}$ is analytic at a.

Example 1. Let α be real and non-zero. Find the singular points of the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{4}$$

and determine which are regular singular points.

The singular points are the zeroes of

$$(1-x^2) = (1-x)(1+x)$$

and are clearly, $x = 1,\ x = -1$.

To test whether 1 is regular, we divide by the function $(1-x^2)$ obtaining

$$y'' - \frac{2x}{(1-x)(1+x)}y' + \frac{\alpha(\alpha+1)}{(1-x)(1+x)}y = 0$$

We see that the multiplier of y' has the form

$$\frac{1}{x-1}\left(\frac{2x}{1+x}\right)$$

and the multiplier of y has the form

$$\frac{1}{x-1}\left(\frac{-\alpha(\alpha+1)}{1+x}\right)$$

The denominators of the factors in parentheses are not zero at $x = 1$, so they are analytic at $x = 1$. Hence, $x = 1$ is a regular singular point.

A similar test for $x = -1$ shows that it also is a regular singular point.

Example 2.

Determine the singular points of

$$2x(x-2)^2y'' + 3xy' + (x-2)y = 0$$

and test each for regularity.

The singular points are clearly $x = 0,\ x = 2$.

To test for regularity, we again divide by the multiplier of y''.

We get

$$y'' + \frac{3}{2(x-2)^2}y' + \frac{1}{2x(x-2)}y = 0$$

We see that $x = x - 0$ does not occur in the multiplier of y' and occurs to the power one in the multiplier of y. So, $x = 0$ is a *regular singular point.* However, $(x - 2)$ occurs to the power two in the multiplier of y', so it is an *irregular singular point.*

The Frobenius Method

Consider second order linear homogeneous differential equation

$$x^2P(x)y'' + xQ(x)y' + R(x)y = 0 \tag{5}$$

with a regular singular point at $x = 0$.

Consider the Maclaurin series expansions for $P(x), Q(x), R(x)$.

$$P(x) = p_0 + p_1x + p_2x^2 + \dots$$

$$Q(x) = q_0 + q_1x + q_2x^2 + \dots$$

$$R(x) = r_0 + r_1x + r_2x^2 + \dots$$

with $p_0 \neq 0$.

If $P(x), Q(x), R(x)$ were simply constants (i.e., $p_i = q_i = r_i = 0$ for $i > 0$), then the equation (5) would be an Euler equation.

We know that we can find a non-zero solution of the form $y(x) = x^r$ where r is a root of the indicial equation

$$z_0(r) = p_0r(r-1) + q_0r + r_0. \tag{6}$$

Frobenius proved that there always is a non-trivial solution to (6) of the form

$$y(x) = x^{r_1}(1 + \sum_{n\geq 1} c_nx^n) \tag{7}$$

where r_1 is a root of the indicial equation, and the power series $\sum_{n\geq 1} c_nx^n$ has a positive radius of convergence $\rho > 0$.

To say that (7) is a solution of (5) means that the expression $y(x)$ satisfies the differential equation for $0 < x < \rho$.

As we will see, one can get a recurrence relation involving r_1 for the coefficients c_n analogous to what we did for ordinary points.

If r_1 is complex (i.e., of the form $r_1 = \alpha + i\beta$ with $\beta \neq= 0$, then the solution is complex valued and (assuming the P, Q, R are real-valued), we can get two linearly independent solutions by taking the real and imaginary parts of (7).

If both roots of the indicial equation are real, then we take r_1 to be the largest one. We call the solution (7) the *first fundamental solution* of (7).

We can then find a second linearly independent solution by the method of reduction of order.

The form of this latter solution depends on the roots $r_1 \geq r_2$ of (7).

Case 1: $r_1 - r_2$ is not an integer.

In this case a second fundamental solution can also be found of the form

$$y(x) = x^{r_2}(1 + \sum_{n\geq 1} b_n x^n) \tag{8}$$

where the constants b_n can again be gotten from a recurrence relation which now involves r_2.

Case 2: $r_1 = r_2$.

In this case, we assume that the first fundamental solution $y_1(x)$ has already been found, and we get a second fundamental solution of the form

$$y_2(x) = y_1(x)\ ln(x) + x^{r_1} \sum_{n\geq 1} b_n x^n \tag{9}$$

Case 3: $r_1 - r_2 = N$ for some positive integer N.

In this case, we again assume that the first fundamental solution $y_1(x)$ has already been found, and we get a second fundamental solution of the form

$$y_2(x) = a\ y_1(x)\ ln(x) + x^{r_2}(1 + \sum_{n\geq 1} b_n x^n) \tag{10}$$

for some constant a which might turn out to be 0.

In this course we assume that we are in Case 1, and we content ourselves with finding recurrence relations of the coefficients c_n in (7) when P, Q, R are polynomials of degree at most two. This illustrates the main ideas. The general treatment for the other cases will be left to a higher level courses.

Let us write r for the root, set $c_0 \neq 0$, and try to find a recursion method to get the coefficients c_n.

The differential equation now has the form

$$x^2(a + bx + cx^2)y'' + x(d + ex + fx^2)y' + (g + hx + kx^2)y = 0 \tag{11}$$

We seek a solution of the form

$$y(x) = x^r(\sum_{n\geq 0} c_n x^n) = \sum_{n\geq 0} c_n x^{n+r} \tag{12}$$

where $c_0 \neq 0$.

Differentiating term by term and inserting the results into the differential equation gives

$$\begin{array}{c} x^2(a+bx+cx^2)\sum(n+r)(n+r-1)c_n x^{n-2+r} \\ +x(d+ex+fx^2)\sum(n+r)c_n x^{n-1+r} \\ +(g+hx+kx^2)\sum c_n x^{n+r} = 0 \end{array}$$

$$\begin{array}{c} (a+bx+cx^2)\sum(n+r)(n+r-1)c_n x^{n+r} \\ +(d+ex+fx^2)\sum(n+r)c_n x^{n+r} \\ +(g+hx+kx^2)\sum c_n x^{n+r} = 0 \end{array}$$

$$\begin{array}{l} a\ \sum(n+r)(n+r-1)c_n x^{n+r} + b\ \sum(n+r)(n+r-1)c_n x^{n+1+r} \\ +c\ \sum(n+r)(n+r-1)c_n x^{n+2+r} \\ +\ d\sum(n+r)c_n x^{n+r} + e\sum(n+r)c_n x^{n+1+r} + f\ \sum(n+r)c_n x^{n+2+r} \\ +\ g\sum c_n x^{n+r} + h\sum c_n x^{n+1+r} + k\sum c_n x^{n+2+r} = 0 \end{array}$$

Next, we make the powers of x all equal to $n+r$:

$$\begin{array}{l} a\ \sum(n+r)(n+r-1)c_n x^{n+r} + b\ \sum(n+r-1)(n+r-2)c_{n-1} x^{n+r} \\ +c\ \sum(n+r-2)(n+r-3)c_{n-2} x^{n+r} \\ +\ d\sum(n+r)c_n x^{n+r} + e\sum(n+r-1)c_{n-1} x^{n+r} + f\ \sum(n+r-2)c_{n-2} x^{n+r} \\ +\ g\sum c_n x^{n+r} + h\sum c_{n-1} x^{n+r} + k\sum c_{n-2} x^{n+r} = 0 \end{array}$$

Now, the general recurrence relation (denoted GRR) comes from setting the coefficients of the powers of x^{n+r} equal to 0 for each n.

This has the form

$$\begin{array}{l} a\ (n+r)(n+r-1)c_n +\ d(n+r)c_n +\ gc_n \\ +b\ (n+r-1)(n+r-2)c_{n-1} +\ e(n+r-1)c_{n-1} + hc_{n-1} \\ +c\ (n+r-2)(n+r-3)c_{n-2} + f\ (n+r-2)c_{n-2} + kc_{n-2} = 0 \end{array} \tag{13}$$

To express GRR in a convenient way we make some new definitions. Form the matrix C of coefficients of the polynomials P, Q, R.

$$C = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

We call the matrix C the *indicial matrix* of (5).

The indicial polynomials obtained from its columns are called, respectively, the *first, second, third* indicial polynomials of (5).

These are, respectively, the polynomials $z_0(r) = ar(r-1)+dr+g$, $z_1(r) = br(r-1)+er+h$ and $z_2(r) = cr(r-1)+fr+k$.

The *indicial roots* of (5) are the roots of the first indicial polynomial $z_0(r)$.

Then, we can read of the the general recurrence relation for the coefficients c_n of $y(x)$ as

General Recurrence Relation (GRR)

$$c_n z_0(n+r) + c_{n-1} z_1(n-1+r) + c_{n-2} z_2(n-2+r) = 0$$

Examples

Each of the following differential equations has a regular singular point at $x = 0$.

For each of them,

a. Write the indicial matrix C and the first three indicial polynomials $z_0(r), z_1(r), z_2(r)$.

b. Determine the indicial roots $r = r_1, r = r_2$ and write them as $r_1 \geq r_2$ if they are real.

c. If the roots are real, then determine the general recurrence relation for the first fundamental solution

$$y(x) = x^{r_1}(1 + \sum_{n \geq 1} c_n x^n)$$

d. Write the first four recurrence relations of the solution $y(x)$ for $n = 1$, $n = 2$, $n = 3$, $n = 4$

e. Determine the coefficients c_1, c_2, c_3, c_4

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + (x^2 - 1/9)y = 0$

Solution of Problem 1.:
We first write the equation in the standard from by multiplying it by x

$$2x^2y'' + xy' + x^2y = 0$$

The indicial matrix

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Indicial Polynomials: $z_0(r) = 2r(r-1) + r = 2r^2 - r$, $z_1(r) = 0$, $z_2(r) = 1$
Indicial roots: $r_2 = 0, r_1 = \frac{1}{2}$
GRR: For $r = 1/2 : z_0(n + 1/2)c_n + z_1(n - 1 + 1/2)c_{n-1} + z_2(n - 2 + 1/2)c_{n-2} = (2(n + 1/2)^2 - (n + 1/2))c_n + c_0 = 0.$
Recurrence relations:
$n \geq 2$: $(2(n + 1/2)^2 - (n + 1/2))c_n + c_0 = 0.$
$n = 1$: $(2 * 9/4 - 3/2)c_1 = 0, c_1 = 0.$
$n = 2$: $(2 * (5/2)^2 - 5/2)c_2 + c_0 = 0$, $10c_2 + c_0 = 0$, $c_0 = 1$, $c_2 = -1/10.$
$n = 3$: $(2 * (7/2)^2 - 7/2)c_3 + c_1 = 0$, $c_3 = 0.$
$n = 4$: $(2 * 81/4 - 9/2)c_4 + c_2 = 0.$ $36 * c_4 + c_2 = 0$, $c_4 = -1/360.$

Solution of Problem 2.:

$$x^2y'' + xy' + (x^2 - 1/9)y = 0$$

The indicial matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1/9 & 0 & 1 \end{pmatrix}$$

Indicial Polynomials: $z_0(r) = r(r-1) + r - 1/9 = r^2 - 1/9$, $z_1(r) = 0$, $z_2(r) = 1$
Indicial roots: $r_1 = 1/3, r_2 = -1/3$
GRR: For $r = 1/3 : z_0(n + 1/3)c_n + c_{n-2} = 0$

Recurrence relations:
$n \geq 2$: $((n+1/3)^2 - 1/9)c_n + c_{n-2} = 0$.
$n = 1$: $((4/3)^2 - 1/9)c_1 = 0$, $c_1 = 0$.
$n = 2$: $((7/3)^2 - 1/9)c_2 + c_0 = 0$, $((49/9) - 1/9)c_2 + c_0 = 0$, $c_2 = -9/48$.
$n = 3$: $z_0(3 + 1/3)c_3 + c_1 = 0$, $c_3 = 0$.
$n = 4$: $((13/3)^2 - 1/9)c_4 + c_2 = 0$, $(56/3)c_4 + c_2 = 0$, $c_4 = (9/48) * (3/56)$