Name: SOLUTIONS
Math 415, Summer II 2013
Solutions to Quiz #1: 07–10–13.

For each of the following questions, precisely state an argument justifying your reasoning. No calculators or notes are allowed.

1. Determine whether or not the set $W = \{\mathbf{x} \in \mathbb{R}^2 \,|\, x_1 \leq x_2\}$ is a vector space.

This set is not a vector space because it's not closed under scalar multiplication. Consider the scalar $c = -1 \in \mathbb{R}$, and the element $\mathbf{v} = (0, 1)^T \in W$. Then $c\mathbf{v} = (0, -1) \notin W$.

2. Does the set of all polynomials $\mathbf{p}(t) \in \mathbb{R}[t]$ that satisfy $\mathbf{p}(3) = 1$ form a vector space?

No, because it is not closed under addition. Consider $\mathbf{p}, \mathbf{q} \in W$. Then

$$(\mathbf{p} + \mathbf{q})\,(3) = (\mathbf{p})(3) + (\mathbf{q})(3) = 1 + 1 = 2.$$

Therefore, $\mathbf{p} + \mathbf{q} \notin W$.

3. Are the following three vectors

$$\mathbf{x}_1 = (1, 1)^T, \quad \mathbf{x}_2 = (1, 2)^T, \quad \mathbf{x}_3 = (2, 1)^T$$

linearly independent? Do they span $\mathbb{R}^2$? Do they form a basis?

(a) To check for independence, we'll appeal directly to the definition. We look for solutions to

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (1)$$

that can be written as the linear system:

$$\begin{cases} c_1 + c_2 + 2c_3 = 0, \\ c_1 + 2c_2 + c_3 = 0. \end{cases} \quad \longleftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right)$$

Row reducing the augmented system provides,

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right), \qquad (2)$$

which has infinitely many solutions: $(c_1, c_2, c_3)^T = t\,(-3, 1, 1)^T$, for any $t \in \mathbb{R}$.
Setting $t = 1$ provides a solution to (1).
Note that the reduced row echelon form of the matrix defining the linear system *did not* have a pivot for each column! That was the fact that permitted us to find a non-trival solution to (1). (What theorem describes this from the text? What else is equivalent to this?)

(b) Yes, they span $\mathbb{R}^2$. The reduced row echelon form of the matrix defined by the vectors has a pivot in each row, and therefore inserting an arbitrary vector $\mathbf{b}$ into the right hand side of the augmented system always has a solution. (What theorem from our text states this?)
(c) No - vectors form a basis if and only if they are linearly indepenpendent and span the set.

4. Suppose that V is a vector space with a finite basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$. Show that the mapping, $T(\mathbf{v}) : V \to \mathbb{R}^n$ defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is a linear map.
We need to show two things:

$$\begin{cases} T(\mathbf{v}+\mathbf{w}) & = T(\mathbf{v}) + T(\mathbf{w}) \\ T(c\mathbf{v}) & = cT(\mathbf{v}), \end{cases}$$

where c is an arbitrary scalar, and $\mathbf{v}, \mathbf{w} \in V$ are arbitrary vectors.

(a) Let $\mathbf{v}, \mathbf{w} \in V$. Then, there are unique scalars, a_i and c_i such that

$$\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n, \quad \text{and} \quad \mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Therefore,

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= [\mathbf{v}+\mathbf{w}]_{\mathcal{B}} \\ &= [(a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n) + (c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n)]_{\mathcal{B}} \\ &= [(a_1+c_1)\mathbf{b}_1 + \cdots + (a_n+c_n)\mathbf{b}_n]_{\mathcal{B}} \\ &= (a_1+c_1, \ldots, a_n+c_n)^T \\ &= (a_1, \ldots, a_n)^T + (c_1, \ldots, c_n)^T \\ &= [\mathbf{v}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}} \\ &= T(\mathbf{v}) + T(\mathbf{w}) \end{aligned}$$

(b) Likewise, suppose c is a scalar.

$$\begin{aligned} T(c\mathbf{v}) = [c\mathbf{v}]_{\mathcal{B}} &= [c\,(a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n)]_{\mathcal{B}} \\ &= [(ca_1)\mathbf{b}_1 + \cdots + (ca_n)\mathbf{b}_n]_{\mathcal{B}} \\ &= (ca_1, \ldots, ca_n)^T \\ &= c\,(a_1, \ldots, a_n)^T \\ &= c[\mathbf{v}]_{\mathcal{B}} = cT(\mathbf{v}). \end{aligned}$$

Notice how all of the operations in V are identical to those in $\mathbb{R}^n$. This is permitted because we have a *basis* for V.