A COUPLE OF COMMENTS ON LINEAR INDEPENDENCE

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Here are a few notes describing linear independence and how it relates to solving a homogenous linear system. For the relevant notes, see the Appendix and §2.2 in the text. We begin with the definition.

Definition. *We say a collection of vectors* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ *inside a vector space* V *are* linearly independent *if the only solution to*

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_m\mathbf{b}_m = 0 \tag{1}$$

is $c_1 = c_2 = \cdots = c_m = 0$.

Questions about linear independence of a set of vectors (in $\mathbb{R}^n$, and by extension vector spaces with a finite basis) can be answered by looking at solutions to a specific problem. In particular, if define a matrix and vector,

$$A = \begin{pmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \ldots & \mathbf{b_m} \\ | & | & & | \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}, \tag{2}$$

then the c_i's are a solution to (1) if and only if $\mathbf{c}$ is a solution to $A\,\mathbf{c} = 0$.

We can see why this must be the case if we observe what happens when we multiply out the matrix product,

$$A\,\mathbf{c} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_m\mathbf{b}_m = 0. \tag{3}$$

Now, consider the system (3) as an augmented linear system, compactly written as,

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m & 0 \\ | & | & \cdots & | & | \end{array}\right), \tag{4}$$

where the c_i's are the unknowns in the vector $\mathbf{c}$.

Row reduction of this system tells us what solutions exist to (1). More precisely, we can say that a (non-trivial) solution to (1) exists if and only if the reduced row echelon form of A has a pivot in *each*

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column. This tells us whether or not the set of vectors in $\mathcal{B}$ are linearly independent! (c.f. **Theorem 2.3**).

Underdetermined systems. The case of $n < m$ means that we have more vectors than we know what to do with in $\mathbb{R}^n$. Any row reduction will never lead to a pivot in each column because there aren't enough equations to work with.

Example 1. *If the following system reduces to*

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & 0 \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m & 0 \\ | & | & \cdots & | & 0 \end{array}\right) \longrightarrow^{ref} \left(\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{5}$$

then there are infinitely many solutions to $A\,\mathbf{c} = 0$, and therefore the vectors $\mathcal{B}$ are linearly dependent. Note that columns 2 and 4 are free, and only columns 1 and 3 remain as pivot columns.

Overdetermined systems. The case of $n > m$ can lead to any result. Here we don't have enough vectors to span $\mathbb{R}^n$, but our collection may or may not be linearly independent.

Example 2. *If the following system reduces to*

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m & 0 \\ | & | & \cdots & | & | \end{array}\right) \longrightarrow^{ref} \left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{6}$$

then only the trivial solution remains to $A\,\mathbf{c} = 0$, and therefore the vectors $\mathcal{B}$ are linearly independent. Note that each column is a pivot column, and row 4 is trivially satisfied.

Note that the vectors in this example *do not* span $\mathbb{R}^4$. Replacing the right hand side of the augmented system with a vector $\mathbf{b} \in \mathbb{R}^n$ will lead to a contradiction for the last row for some choices of $\mathbf{b}$.

Example 3. *If the following system reduces to*

$$\left(\begin{array}{cccc|c} | & | & \cdots & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m & 0 \\ | & | & \cdots & | & | \end{array}\right) \longrightarrow^{ref} \left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{7}$$

then there are infinitely many solutions to $A\,\mathbf{c} = 0$, and therefore the vectors $\mathcal{B}$ are linearly dependent. Note that only columns 1 and 3 remain as pivot columns, and column 2 is free.

Square matrices. In the case of a square matrix, (i.e. $n = m$ vectors sitting in $\mathbb{R}^n$), then there are fewer choices to play with. If it row reduces to the identity matrix, then we can say that the matrix is invertible, and therefore say much more about the vectors. See **Theorem 2.5** for the complete laundry list.

Example 4. *If the following (square) system reduces to*

$$\left(\begin{array}{ccc|c} | & | & | & 0 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & 0 \\ | & | & | & 0 \end{array}\right) \longrightarrow^{ref} \left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \end{array}\right), \tag{8}$$

then there is a unique (trivial) solution to $A\,\mathbf{c} = 0$*, and therefore the vectors* $\mathcal{B}$ *are linearly independent. Note that each column remains as a pivot column.*

You can also check for independence by looking at the determinant of A. See **Theorem 2.5** for other items you can consider.

Example 5. *If the following (square) system reduces to*

$$\left(\begin{array}{ccc|c} | & | & | & 0 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & 0 \\ | & | & | & 0 \end{array}\right) \longrightarrow^{ref} \left(\begin{array}{ccc|c} 1 & 2 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow^{rref} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{9}$$

then there are infinitely many solutions to $A\,\mathbf{c} = 0$*, and therefore the set* $\mathcal{B}$ *is linearly dependent.*

In particular, setting $c_2 = t \in \mathbb{R}$ *(the free column), we have a solution,*

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

The combination, $-2\mathbf{b}_1 + \mathbf{b}_2 = 0$ *will then a non-trivial solution to* (1).