HOW TO COMPUTE POWERS OF A MATRIX WITH COMPLEX EIGENVALUES

DAVID SEAL

We consider the discrete time evolution problem,

$$\mathbf{x}(n) = A\mathbf{x}(n-1), \quad A \in M_{m,m}, \tag{1}$$

with $\mathbf{x}(0)$ prescribed. The exact solution is

$$\mathbf{x}(n) = A^n \mathbf{x}(0), \tag{2}$$

which is difficult to write down a closed form solution to. If A has a complete set of eigenvalues, $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$, with corresponding eigenvectors $\{\lambda_1, \dots, \lambda_m\}$, then we can simply track the coefficients of these eigenvectors at each time level[1]:

$$\mathbf{x}(n) = a_1(n)\mathbf{b}_1 + \cdots a_m(n)\mathbf{b}_m. \tag{3}$$

Given this basis, the exact solution (2) can then be written as[2]:

$$\mathbf{x}(n) = \lambda_1^n a_1(0)\mathbf{b}_1 + \cdots \lambda_m^n a_m(0)\mathbf{b}_m. \tag{4}$$

Example 1. *Consider the matrix*

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{5}$$

with initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}. \tag{6}$$

Compute $\mathbf{x}(n)$ *if* $\mathbf{x}$ *satisfies the recurrance relation in* (1).

For this problem, we need to compute the following three items:

(1) Eigenvalues λ_i of A.
(2) Eigenvectors $\mathbf{b}_i$ of A.

Date: July 26, 2013.

[1] Two questions to think about:

(1) Why do a set of such coefficients a_i exist?
(2) Why are those coefficients unique?

[2] Why? see if you can show this!

(3) The initial conditions, written in this basis of eigenvectors: $[\mathbf{x}(0)]_{\mathcal{B}} = (a_1(0), a_2(0))^T$. Note that this representation comes from a clever observation, or more generally, from using the change of basis matrix:

$$\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = [\mathbf{x}(0)]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}\mathbf{x}(0) = P_{\mathcal{E}\mathcal{B}}^{-1}\mathbf{x}(0). \tag{7}$$

We start by computing the charactistic polynomial, then we'll compute the eigenvalues, then we can compute the eigenvectors. After doing that, we can compute the change of basis matrix, and write down our complete solution.

$$p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \lambda - \frac{1}{\sqrt{2}} \end{vmatrix} = \left(\lambda - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}. \tag{8}$$

The roots of this are

$$\lambda = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}, \tag{9}$$

which can be expressed in polar coordinates,

$$\lambda = e^{\pm i\pi/4}. \tag{10}$$

The corresponding eigenvectors are $\mathbf{b}_1 = (1, -i)^T$ and $\mathbf{b}_2 = (1, i)^T$. The change of bases matrices are:

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \qquad P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{11}$$

This gives us the coefficients, $a_1(0)$ and $a_2(0)$:

$$[\mathbf{x}(0)]_{\mathcal{B}} = \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = P_{\mathcal{B}\mathcal{E}}\,\mathbf{x}(0) = \frac{5}{2}\begin{pmatrix} e^{i\beta} \\ e^{-i\beta} \end{pmatrix}, \tag{12}$$

where $\beta = \tan^{-1}(4/3)$.

Now that we have all the λ_i, the $\mathbf{b}_i$ and the initial conditions, we can write the final solution in (2) as

$$\mathbf{x}(t) = \frac{5}{2}e^{\frac{in\pi}{4}}e^{i\beta}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{5}{2}e^{-\frac{in\pi}{4}}e^{-i\beta}\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{13}$$

At this point, it's not obvious that this is a real-valued vector, which it indeed is. To see why, let's collect like terms:

$$\mathbf{x}(t) = \frac{5}{2}\begin{pmatrix} e^{i\left(\beta + \frac{\pi n}{4}\right)} + e^{-i\left(\beta + \frac{\pi n}{4}\right)} \\ -i\left(e^{i\left(\beta + \frac{\pi n}{4}\right)} - e^{-i\left(\beta + \frac{\pi n}{4}\right)}\right) \end{pmatrix} = 5\begin{pmatrix} \frac{e^{i\left(\beta + \frac{\pi n}{4}\right)} + e^{-i\left(\beta + \frac{\pi n}{4}\right)}}{2} \\ \frac{e^{i\left(\beta + \frac{\pi n}{4}\right)} - e^{-i\left(\beta + \frac{\pi n}{4}\right)}}{2i} \end{pmatrix}. \tag{14}$$

The final observation to make is that Euler's identity can be inverted:

$$\begin{cases} e^{i\theta} = \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} = \cos(\theta) - i\sin(\theta) \end{cases} \implies \begin{cases} \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos(\theta); \\ \frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin(\theta). \end{cases} \tag{15}$$

Plugging this identity into the above equation with $\theta = \beta + \frac{\pi n}{4}$ yields:

$$\mathbf{x}(t) = 5\begin{pmatrix} \cos(\alpha + \frac{n\pi}{4}) \\ \sin(\alpha + \frac{n\pi}{4}) \end{pmatrix}. \tag{16}$$

You can check that this solution is indeed correct, by computing this solution for small values of n. In MATLAB, or a similar program, (or programming *language*, such as Python with the numpy extension),

```
>> A = 1/sqrt(2) * [1 -1; 1 1];
>> alpha = atan( 4 / 3 );
>> x0 = [ 3; 4];
>> n = 10;
>> xex = A^n * x0

xex =

   -4.0000
    3.0000

>> xtst = 5 * [ cos( alpha + (n*pi)/4) ; sin( alpha + (n*pi)/ 4 ) ]

xtst =

   -4.0000
    3.0000
```