HOMEWORK SET 1

MATH 415, SUMMER 2013

This is the first set of homework set #1. You are *strongly encouraged* to work out *all* of the problems. Our first quiz will be on next Wednesday, July 10^{th}, and will cover material based on these problems.

1. Problems

A handful of these problems are taken from Sadun's book, with minor changes to the wording.

(1) Suppose a sequence of vectors, $\mathbf{x}(n)$ is defined recursively through

$$\mathbf{x}(n) = A\mathbf{x}(n-1), \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Find all solutions to $\mathbf{x}(n)$ with prescribed initial conditions, $\mathbf{x}(0)$. Which component of $\mathbf{x}$ grows the fastest?

(2) Solve the first-order linear differential equation,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

with initial conditions prescribed by $\mathbf{x}(0) = (1,1)^T$.
Hint: Use the transformation

$$y_1 = (x_1 + x_2)/2, \quad y_2 = (x_1 - x_2)/2.$$

(3) Determine whether or not the set $W = \{\mathbf{x} \in \mathbb{R}^2 \,|\, x_1 \leq x_2\}$ is a vector space. That is, is W a sub-space of the vector space $\mathbb{R}^2$?

(4) Determine whether or not the set,

$$W = \left\{ c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} \in \mathbb{R}^3 \,|c_1, c_2 \in \mathbb{R} \right\}$$

is a vector space.

(5) Show that the additive inverse of $\mathbf{x}$ is the same as -1 times $\mathbf{x}$ (thereby, justifying the notation, $-\mathbf{x}$).

(6) Find 3 qualitatively different examples of maps between vector spaces that are not linear transformations.

(7) Let I be indefinite integration on $C[0,1]$: $(If)(x) = \int_0^x f(t)dt$. Show that I is a linear operator.

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(8) Show that the mapping, $P : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $P(x, y, z) = (x, y)$ is a linear transformation. This is called the *projection* operator.

(9) Prove that the image,

$$T(V) = \{\mathbf{y} \,|\, T(\mathbf{x}) = \mathbf{y} \text{ for some } \mathbf{x} \in V\}$$

of a linear transformation $T : V \to W$ is a subspace of W.

(10) Are the vectors $\mathbf{b}_1 = (r, 0)^T$ and $\mathbf{b}_2 = (0, s)^T$ with $r, s \neq 0$ linearly independent?

(11) Are the vectors $\mathbf{b}_1 = (1, 1, -2)^T$ and $\mathbf{b}_2 = (1, -2, 1)^T$ linearly independent?

(12) Are the vectors $\mathbf{b}_1 = (1, 1, 1)^T$, $\mathbf{b}_2 = (1, 2, 3)^T$ and $\mathbf{b}_3 = (1, 4, 9)^T$ linearly independent?

(13) Are the columns of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

linearly independent?

(14) Are the columns of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

linearly independent?

(15) Let V be a vector space with basis $\mathcal{B}$. If $\mathbf{x}, \mathbf{y} \in V$ and c is a scalar, then

$$[\mathbf{x} + \mathbf{y}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{B}} + [\mathbf{y}]_{\mathcal{B}}, \quad \text{and} \quad [c\mathbf{x}]_{\mathcal{B}} = c[\mathbf{x}]_{\mathcal{B}}.$$

(Hence, the mapping, $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is a linear mapping. See also, theorem 2.7 from the text.)

(16) If $V = \mathbb{R}^n$ and $\mathcal{B} = \mathcal{E}$, is the standard basis, show that $[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

(17) Determine which of the following sets of vectors are linearly independent, which span, and which are bases.

   (a) In $\mathbb{R}_2[t]$, $\mathbf{b}_1 = 1 + t + t^2, \mathbf{b}_2 = 1 + 2t + 3t^2$, $\mathbf{b}_3 = 1 + 4t + 9t^2$. How does this compare to Exercise 12?

   (b) In $M_{2,2}$, $\mathbf{b}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(18) Let V be the subset of $M_{2,2}$ consisting of symmetric matrics ($A = A^T$).

   (a) Show that V is a subspace, i.e. V is a vector space in its own right.

   (b) Find a basis for V.

   (c) What is the dimension of V.

(19) If L is an isomorphism from V to W, show that L^{-1} is well defined and is an isomorphism from W to V.

2. A couple of theorems

I incorrectly stated a theorem from section. The correct statement is,

Theorem. *Let A be an $n \times m$ matrix. The the following statements are equivalent:*

(1) *The columns of A span $\mathbb{R}^n$*

(2) *The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*

(3) *The reduced row-echelon form of the matrix A has a pivot in each row.*

The correct results for *square* matrices is given by a longer theorem.

Theorem. *Let A be an $n \times n$ matrix. Then the following statements are equivalent:*

(1) *The columns of A are linearly independent.*
(2) *The columns of A span $\mathbb{R}^n$.*
(3) *The columns of A for a basis for $\mathbb{R}^n$.*
(4) *The equation $A\mathbf{x} = \mathbf{b}$ has a* unique *solution for every $\mathbf{b} \in \mathbb{R}^n$.*
(5) *A is an invertible matrix.*
(6) *The determinant of A is nonzero.*
(7) *A is row equivalent to the identity matrix.*