Midterm Exam

Math 415
Tuesday, July 30^{th}, 2013

Name: Key

Question	Points	Your Score
Q1	10	
Q2	15	
Q3	10	
Q4	10	
Q5	15	
Q6	20	
Q7	20	
TOTAL	100	

Read all of the following information before starting the exam:

- *One* 8-1/2" by 11" sheet of personal notes is allowed for use during the exam.
- Show all work, clearly and in order. "Answers" without justification will receive zero credit.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: `https://www.msu.edu/~ombud/academic-integrity/index.html`. Simply put, don't cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!

Problem 1 (10 points) Are the following sets vector subspaces?

(a) The set of all elements $(x, y) \in \mathbb{R}^2$ that satisfy $x - 2y = 0$.

Yes. $T(x) = Ax$ with $A = [1 \ -2]$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$. This set can be identified as $\mathrm{Ker}(T)$.

(b) The set of all polynomials in $\mathbb{R}[t]$ with degree exactly 2.

No. Consider $p(t) = t^2 + 1$ and $w(t) = t^2$ both in this set.

$p + (-1) \cdot w = t^2 + 1 - t^2 = 1$ is of degree 0, and therefore this set is not closed under addition.

Problem 2 (15 points) Suppose V is a vector space with finite basis $\mathcal{B}$. Show that the mapping $T : V \to \mathbb{R}^n$ defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is

(a) Well defined.

(b) Linear.

Clearly indicate which assumptions are used where in your write-up.

See notes from lecture and the write-up posted on the website. See solns to Quiz #1 for part (b).

Problem 3 (10 points) For which choices of the constant k do the following vectors form a basis for $\mathbb{R}^3$?

$$\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}2\\3\\k\end{pmatrix}.$$

Consider $A = \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ 2 & 3 & k\end{pmatrix}$.

$$\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ 2 & 3 & k\end{pmatrix} \xrightarrow{R_3 = -2R_1 + R_3} \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ 0 & 3 & -4+k\end{pmatrix}$$

$$\xrightarrow{-3R_2 + R_3} \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ 0 & 0 & -13+k\end{pmatrix}$$

$\therefore$ vectors are linearly indep. iff $k \neq 13$.

This is equivalent to forming a basis b/c there are exactly 3 vectors.

Problem 4 (10 points) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear operator with a complete set of eigenvectors:

$$\mathcal{B}\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\},$$

and corresponding eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$. For $\mathbf{v} = (1, 2, -1, -4)^T$, compute $T^{2013}(\mathbf{v})$. That is, determine the result of applying T to $\mathbf{v}$ 2013 times.

By inspection, we can construct the correct coefficients of the $\vec{b}_i$:

$\cancel{\vec{b}_1 + 0\vec{b}_2} \neq \cancel{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}$ We can also directly find these.

We Want: $a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + a_4\vec{b}_4 = \begin{pmatrix}1\\2\\-1\\-4\end{pmatrix}$,

~~Obviously $a_1 = 1$, and $a_2 = 0$.~~

Which can be found through row operations

$$\left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 3 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -4 \end{array}\right) \xrightarrow{\text{use } R_1} \left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & -4 \\ 0 & 3 & 2 & 1 & -8 \end{array}\right)$$

$$\xrightarrow{\text{use } R_2} \left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 2 & 1 & -8 \end{array}\right) \xrightarrow{R_4 = -2R_3 + R_4} \left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$\therefore \vec{v} = \vec{b}_1 + (-4)\vec{b}_3$, so $T^{2013}\vec{v} = 1^{2013}\vec{b}_1 + (-4)3^{2013}\vec{b}_3$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 + (-4)3^{2013} \\ 4 + (-4)7^{2013} \end{pmatrix}$$

Problem 5 (15 points) Suppose T is a linear operator on a vector space V.

(a) Show that the set $E_\lambda = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda \mathbf{v}\}$ is a vector space.

<u>Scalar</u> <u>Multiplication</u>

Let $c \in \mathbb{R}$ and $v \in E_\lambda$.

$T(c \cdot v) = c \cdot T(v) = c \cdot (\lambda v) = \lambda(c \cdot v)$.

$\therefore$ E_λ is closed under scalar multiplication.

<u>Additive</u> <u>Closure</u>

Let $v, w \in E_\lambda$.

$T(v+w) = T(v) + T(w) = \lambda v + \lambda w = \lambda(v+w)$

$\therefore$ E_λ is closed under vector addition.

B/c E_λ is a subset of a vector space, these two properties imply E_λ is a vector space.

(b) Suppose $\mathbf{v}_1 = (1,2,3)^T$ and $\mathbf{v}_2 = (3,0,0)^T$ are eigenvalues of T with eigenvalue $\lambda = -2$. Compute $T\left((11,4,6)^T\right)$.

First, note that $\vec{v} = \begin{pmatrix} 11 \\ 4 \\ 6 \end{pmatrix} = 2 \cdot \vec{v}_1 + 3 \cdot \vec{v}_2$

and therefore $\vec{v} \in E_{-2}$ by part (a).

$\therefore$ $T(\vec{v}) = (-2) \cdot \vec{v} = \begin{pmatrix} -22 \\ -8 \\ -12 \end{pmatrix}$.

Problem 6 (20 points) Consider the linear transformation $T : \mathbb{R}_3[t] \to \mathbb{R}_1[t]$ defined by

$$T(\mathbf{p})(t) = \mathbf{p}(1) - 2\mathbf{p}(2)t.$$

Compute the matrix of the linear transformation, and find a basis for the kernel and range. Specify your choice of basis vectors.

We'll use std. basis: $\mathcal{E}_3 = \{1, t, t^2\}$

and $\mathcal{E}_1 = \{1, t\}$.

$$T(\vec{e}_1) = 1 - 2 \cdot t$$

$$T(\vec{e}_2) = 1 - 2 \cdot 2 \cdot t = 1 - 4t$$

$$T(\vec{e}_3) = 1 - 2 \cdot 2^2 t = 1 - 8t.$$

$$A = [T]_{\mathcal{E}_1 \mathcal{E}_3} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -4 & -8 \end{bmatrix}.$$ is the matrix of the lin. transf.

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -4 & -8 \end{pmatrix} \xrightarrow{R_2 = 2R_1 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 = -R_2 + R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Basis for range: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ (i.e. $\{1, t\}$)

~~Bas~~ Basis for kernel:

$x_1 = 2x_3$ $\quad \vec{x} \in \ker(A)$ iff $\vec{x} = s \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$x_2 = -3x_3$

$x_3 = x_3$ $\quad$ So basis is given by $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}$, i.e. $\{2 - 3t + t^2\}$

Note: $T(2-3t+t^2)(t) = (2-3+1) - 2 \cdot (2-6+4)t$

Problem 7 (20 points) Consider the recurrance relation,

$$\mathbf{x}(n) = A\mathbf{x}(n-1), \qquad A = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

Compute $\mathbf{x}(n)$ for all n, and compute $\lim_{n\to\infty} \mathbf{x}(n)$.

Soln. is given by $\vec{X}(n) = \lambda_1^n a_1(0)\vec{b}_1 + \lambda_2^n a_2(0)\vec{b}_2$.

Let's diagonalize A!

1.) Characteristic polynomial

$$\lambda I - A = \begin{pmatrix} \lambda - 1/2 & -3/4 \\ -1/2 & \lambda - 1/4 \end{pmatrix}. \qquad \det(\lambda I - A) = (\lambda - 1/2)(\lambda - 1/4) - 3/8$$

$$P_A(\lambda) = \lambda^2 - \left(\tfrac{1}{2} + \tfrac{1}{4}\right)\lambda + \tfrac{1}{8} - \tfrac{3}{8}$$

$$= \lambda^2 - \tfrac{3}{4}\lambda - \tfrac{2}{8} \qquad \boxed{P_A(\lambda) = \lambda^2 - \tfrac{3}{4}\lambda - \tfrac{1}{4}}$$

2.) Solve for eigenvalues

$$P_A(\lambda) = 0$$

$$= \lambda^2 - \tfrac{3}{4}\lambda - \tfrac{1}{4}$$

$$\lambda = \frac{3/4 \pm \sqrt{9/16 + 1}}{2} = \frac{1}{2}\left(\frac{3}{4} \pm \sqrt{\frac{9+16}{16}}\right)$$

$$= \frac{1}{2}\left(\frac{3}{4} \pm \frac{\sqrt{25}}{\sqrt{16}}\right) = \frac{1}{2}\left(\frac{3}{4} \pm \frac{5}{4}\right) = \frac{3}{8} \pm \frac{5}{8}$$

$$\boxed{\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -\tfrac{1}{4}}$$

3.) Solve for eigenvectors

$\underline{\lambda_1 = 1:}$

$$\lambda I - A = \begin{pmatrix} 1/2 & -3/4 \\ -1/2 & 3/4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$$

$X_2 = s$, $2X_1 = 3s$, $X_1 = \frac{3}{2}s$.

$v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ OR rescaled, $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

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Recall $\lambda I - A = \begin{pmatrix} \lambda - 1/2 & -3/4 \\ -1/2 & \lambda - 1/4 \end{pmatrix}$

$\underline{\lambda_2 = -1/4:}$

$$-\tfrac{1}{4} I - A = \begin{pmatrix} -3/4 & -3/4 \\ -1/2 & -1/2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\boxed{\begin{array}{l} \lambda = 1: \quad \text{eigenvector} \quad \vec{b}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ \lambda = -1/4 \quad \text{eigenvector} \quad \vec{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}}$$

We now need $a_1(0)$ and $a_2(0)$. I.e., we need

$$\vec{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix} = a_1(0) \begin{pmatrix} 3 \\ 2 \end{pmatrix} + a_2(0) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 1 & 100 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow[-3R_2]{2R_1} \left(\begin{array}{cc|c} 6 & 2 & 200 \\ -6 & 3 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 6 & 2 & 200 \\ 0 & 5 & 200 \end{array}\right)$$

$$\xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 3 & 1 & 100 \\ 0 & 1 & 40 \end{array}\right) \qquad \begin{array}{l} 3a_1 + 40 = 100, \text{ so } a_1 = \frac{60}{3} = 20. \\ a_2 = 40 \end{array}$$

$$\vec{x}(0) = 20 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 40 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\vec{x}(n) = 20 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 40 \left(-1/4\right)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\boxed{\lim_{n \to \infty} \vec{x}(n) = \begin{pmatrix} 60 \\ 40 \end{pmatrix}}$$

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