Math 421	Homework 9	**Assigned**:	Monday, April 14th
Spring 2014	"Partial derivatives and partial integration"	**Due**:	Friday, April 18th

Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

WARMUP PROBLEMS (Not to be turned in)

1. True or False. If $f : E \to \mathbb{R}^m$ is uniformly continuous, then E is compact.
2. True or False. If E is not open, then E is closed.
3. True or False. If $f : E \to \mathbb{R}^m$ is continuous, then f is uniformly continuous (on E) if and only if E is compact.
4. If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, prove that $F(y) = \int_a^b f(x, y)\, dx$ is continuous on $[c, d]$.

HOMEWORK EXERCISES

1. Suppose $H = [a, b] \times [c, d]$ is a rectangle, and $k : H \to \mathbb{R}$ is continuous, and $g : [a, b] \to \mathbb{R}$ is integrable. Prove that
$$F(y) = \int_a^b g(x)k(x, y)\, dx$$
is uniformly continuous on $[a, b]$. *Hint:* you may use the fact that H is compact. Such a k is oftentimes called a "kernel," and is commonplace when constructing integral solutions to differential equations.

2. Construct all of the partial derivatives of
$$f(x, y, z) = \left(\frac{x}{y}, \sin\left(xz - 2\pi y\right), \tan^{-1}\left(z^2\right)\right)^T$$
Note that all of the single variable rules (product rule, chain rule, etc.) carry over to partial derivatives when you consider freezing every other variable.

3. Compute all mixed second-order partial derivatives of the following function, and verify that the mixed partial derivatives are equal:
   (a) $f(x, y) = xe^y$.
   (b) $f(x, y) = \cos(xy)$.

4. If $A \subseteq \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}$ obtains a max (or minumum) at $\mathbf{x} = \mathbf{a} \in A$ and $\frac{\partial f}{\partial x_j}(\mathbf{a})$ exists for each j, prove that $\frac{\partial f}{\partial x_j} = 0$. Is the converse true?

5. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Find the partial derivatives of the following functions

   (a) $f(x, y) = \int_a^{x+y} g(s)\, ds$.

   (b) $f(x, y) = \int_x^y g(s)\, ds$.

6. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *independent of the second variable* if for each $x \in \mathbb{R}$, we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that f is independent of the second variable if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$. What is $\frac{\partial f}{\partial x}$ in terms of g?

7. If $f : \mathbb{R}^2 \to \mathbb{R}$, and $\frac{\partial f}{\partial x_2} = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, show $f(x_1, x_2) = g(x_1)$ for some g. That is, show that f is independent of the second variable. If $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$, show that f is constant.

8. For each of the following functions f, find the matrix representation of a linear transformation $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - T(h)\|}{h} = 0.$$

   (a) $f(x) = (x^2, \sin(x))$.

   (b) $f(x) = \left(e^x, x^{1/3}, 1 - x^2\right)$.

9. Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and recall the definition of the *operator norm*,

$$\|T\| := \sup_{\mathbf{x} \neq 0} \frac{\|T(\mathbf{x})\|}{\|\mathbf{x}\|}.$$

   (a) Show that the supremum need only be taken over the unit sphere. That is, prove that $\sup_{\|\mathbf{x}\|=1} \|T(\mathbf{x})\| = \|T\|$.

   (b) Define

$$m := \inf \{C > 0 : \|T(\mathbf{x})\| \leq C\|\mathbf{x}\| \quad \text{for all } \mathbf{x}\}.$$

   Prove that $m = \|T\|$.