Math 421	Homework 8	**Assigned**:	Thursday, March 27
Spring 2014	"Continuity and limits: Part II"	**Due**:	Wednesday, April 2

Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

WARMUP PROBLEMS (Not to be turned in)

1. Compare and contrast the definitions of a *limit*, and *continuity* at a point. Specifically, what sort of domains do we consider when looking at continuity vs. limits?

HOMEWORK EXERCISES

1. [**The sequential characterization of limits of functions**] Suppose $f : V\backslash\{\mathbf{a}\} \to \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n$, and $\mathbf{a} \in V$. Prove that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if an only if $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ for every sequence $\mathbf{x}_k \in V\backslash\{\mathbf{a}\}$ that satisfies $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$.

2. [**The sequential charactization of continuity**] Suppose $f : E \to \mathbb{R}^m$. Prove that f is continuous at $\mathbf{a} \in E$ if an only if $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{a})$ for every sequence $\mathbf{x}_k \in E$ that satisfies $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$. Compare with the results of problem 1.

3. Suppose $f : E \to \mathbb{R}^m$, where $E \subseteq \mathbb{R}^n$ is a closed set. Prove that f is continuous on E if and only if $f^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}^m$.

4. Consider $f, g : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sin(x)$, and $g(x) = x/|x|$ if $x \neq 0$, and $g(0) = 0$.

   (a) Define $E_1 = (0, \pi)$, $E_2 = [0, \pi]$, $E_3 = (-1, 1)$ and $E_4 = [-1, 1]$. For $j = 1, \ldots 4$, compute $f(E_j)$ and $g(E_j)$. What conclusions can you draw about the images of connected/closed/open sets?

   (b) Define $F_1 = (0, 1)$, $F_2 = [0, 1]$, $F_3 = (-1, 1)$ and $F_4 = [-1, 1]$. For $j = 1, \ldots 4$, compute $f^{-1}(F_j)$ and $g^{-1}(F_j)$. What conclusions can you draw about the inverse images of connected/closed/open sets?

5. Let H be a non-empy, compact subset of $\mathbb{R}^n$.

   (a) If $f : H \to \mathbb{R}^m$ is a function, we define

   $$\|f\|_H := \sup\{\|f(\mathbf{x})\| : \mathbf{x} \in H\}.$$

   Show that if f is continuous, then there exists an $\mathbf{x}^* \in H$ such that $\|f\|_H = \|f(\mathbf{x}^*)\|$.

   (b) Consider the following definition, which is the multi-variable extension of what you have already seen in the single variable case.

Definition 1 *We say a sequence of functions* $f_k : H \to \mathbb{R}^m$ converge uniformly *to* $f : H \to \mathbb{R}^n$ *if for every* $\epsilon > 0$*, there exists an* $N \in \mathbb{Z}_{\geq 1}$ *such that for every* $\mathbf{x} \in H$ *and* $k \geq N$*, we have* $\|f_k(\mathbf{x}) - f(\mathbf{x})\| < \epsilon$*.*

Show that $\|f_k - f\|_H \to 0$ if and only if f_k converge to f uniformly.

6. Show that $\|f_k - f\|_H \to 0$ if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that $k, j \geq N$ implies $\|f_k - f_j\|_H < \epsilon$.

7. Using the new topological tools that we have, construct a shorter proof of the following theorem, which appeared in homework 1: If $f \geq 0$ is continuous on $[a, b]$, prove that $\int_a^b f(x)\,dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

8. Suppose $H = [a, b] \times [c, d]$ is a rectangle, and $f : H \to \mathbb{R}$ is continuous, and $g : [a, b] \to \mathbb{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x) f(x, y)\,dx$$

is uniformly continuous on $[a, b]$. *Hint:* you may use the fact that H is compact.