Math 421	Homework 7	**Assigned**:	Tuesday, March 18
Spring 2014	"Limits and Continuity"	**Due**:	Monday, March 24

Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

WARMUP PROBLEMS (Not to be turned in)

1. Suppose $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ are sequences in $\mathbb{R}^n$ that satisfy $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{y}_k\| = 0$.
   (a) Does $\lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty} \mathbf{y}_k$?
   (b) If both sequences are bounded, does $\lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty} \mathbf{y}_k$?
   (c) Prove that $\lim_{k_j\to\infty} \mathbf{x}_{k_j} = \lim_{k_j\to\infty} \mathbf{y}_{k_j}$ for some subsequence.
2. True or False. If f is continuous at $\mathbf{a} \in \mathbb{R}^n$, does $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$?

HOMEWORK EXERCISES

1. Suppose $f : V\backslash\{\mathbf{a}\} \to \mathbb{R}^m$. Prove that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if an only if there exists an open set $U \subseteq V \subseteq \mathbb{R}^n$ with $\mathbf{a} \in U$ such that $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ for every sequence $\mathbf{x}_k \in U\backslash\{\mathbf{a}\}$ that satisfies $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$.
2. Prove that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists, where $f : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}^2$ is defined by
$$f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}.$$
3. Prove that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists, where $f : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}^2$ is defined by
$$f(x, y) = \frac{|x|^\alpha y^4}{x^2 + y^4},$$
and $\alpha > 0$ is a fixed positive number. Does the same limit exist if $\alpha = 0$?
4. Suppose $f : E \to \mathbb{R}^m$. Prove that f is continuous at $\mathbf{a}$ if an only if there exists a relatively open set $U \subseteq \mathbb{R}^n$ with $\mathbf{a} \in U$ such that $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{a})$ for every sequence $\mathbf{x}_k \in U$ that satisfies $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$. Compare with the results of problem 1.
5. Suppose $f : X \to Y$ is a function, where X and Y are arbitrary sets. Prove or disprove the following identities.

(a) If $A_\alpha \subseteq Y$, $\alpha \in \mathcal{A}$ is a collection of sets, then

$$\bigcup_{\alpha \in \mathcal{A}} f^{-1}(A_\alpha) = f^{-1}\left(\bigcup_{\alpha \in \mathcal{A}} A_\alpha\right).$$

(b) If $B_\beta \subseteq X$, $\beta \in \mathcal{B}$ is a collection of sets, then

$$\bigcup_{\beta \in \mathcal{B}} f(B_\beta) = f\left(\bigcup_{\beta \in \mathcal{B}} B_\beta\right).$$

If either of these are not true, prove set inclusion if it exists.

6. Suppose $f : X \to Y$ is a function, where X and Y are arbitrary sets. Prove or disprove the following identities.

   (a) If $A_\alpha \subseteq Y$, $\alpha \in \mathcal{A}$ is a collection of sets, then

   $$\bigcap_{\alpha \in \mathcal{A}} f^{-1}(A_\alpha) = f^{-1}\left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha\right).$$

   (b) If $B_\beta \subseteq X$, $\beta \in \mathcal{B}$ is a collection of sets, then

   $$\bigcap_{\beta \in \mathcal{B}} f(B_\beta) = f\left(\bigcap_{\beta \in \mathcal{B}} B_\beta\right).$$

   If either of these are not true, prove set inclusion if it exists.

7. Consider $f, g : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sin(x)$, and $g(x) = x/|x|$ if $x \neq 0$, and $g(0) = 0$.

   (a) Define $E_1 = (0, \pi)$, $E_2 = [0, \pi]$, $E_3 = (-1, 1)$ and $E_4 = [-1, 1]$. For $j = 1, \ldots 4$, compute $f(E_j)$ and $g(E_j)$. What conclusions can you draw about the images of closed/open sets?

   (b) Define $F_1 = (0, 1)$, $F_2 = [0, 1]$, $F_3 = (-1, 1)$ and $F_4 = [-1, 1]$. For $j = 1, \ldots 4$, compute $f^{-1}(F_j)$ and $g^{-1}(F_j)$. What conclusions can you draw about the inverse images of closed/open sets?

8. Suppose $A \subseteq \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}^m$. Prove that f is continuos on A if and only if $f^{-1}(V)$ is open for every open set $V \subseteq \mathbb{R}^m$. What's the difference between this theorem and Thm. 9.26?

9. The problem is an extension of the previous problem. It says that we need only concern ourselves with the inverse image of basis elements that *generate* the topology of $\mathbb{R}^m$. Suppose $A \subseteq \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}^m$. Prove that f is continuos on A if and only if $f^{-1}(B_\epsilon(\mathbf{x}))$ is open for every point $\mathbf{x} \in \mathbb{R}^m$ and $\epsilon > 0$.