Math 421	Homework 6	**Assigned**:	Monday, March 10
Spring 2014	"Heine-Borel and Limits"	**Due**:	Friday, March 14

Directions:

- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

HOMEWORK EXERCISES

1. Suppose K is compact in $\mathbb{R}^n$, and $E \subseteq K$. Prove that E is compact if and only if E is closed.

2. Suppose K is compact in $\mathbb{R}^n$, and for every $\mathbf{x} \in K$, there is an $r = r(\mathbf{x}) > 0$ such that $B_r(\mathbf{x}) \cap K = \{\mathbf{x}\}$. Prove that K is a finite set.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, and $K \subseteq \mathbb{R}^n$ is compact and connected. For each $\mathbf{x}$, suppose there exists a $\delta = \delta(\mathbf{x}) > 0$ such that $f(\mathbf{x}) = f(\mathbf{y})$ for all $\mathbf{y} \in B_{\delta(\mathbf{x})}(\mathbf{x})$. Prove that f is constant on K.

   Note: This is an excellent example of a "local" to "global" result[1].

4. Recall that we defined the distance between a point $\mathbf{x} \in \mathbb{R}^n$ and a set $A \subseteq \mathbb{R}^n$ as

   $$d(A, \mathbf{x}) := \inf \{\|\mathbf{x} - \mathbf{y}\| \,:\, \mathbf{y} \in A\}$$

   Define the distance between two sets $A, B \subseteq \mathbb{R}^n$ as

   $$d(A, B) := \inf \{\|\mathbf{x} - \mathbf{y}\| \,:\, \mathbf{x} \in A,\, \mathbf{y} \in B\}.$$

   (a) Prove that if A and B are compact sets that satisfy $A \cap B = \emptyset$, then $d(A, B) > 0$.

   (b) Show that there exist nonempty, closed sets $A, B \subset \mathbb{R}^2$ such that $A \cap B = \emptyset$, but $d(A, B) = 0$.

5. Consider the function

   $$f(x, y) = \left(\frac{x-1}{y-1},\, x+2\right),$$

   whose domain is yet to be determined.

   (a) Identify the co-domain[2] of f.

   (b) Find the largest domain E, where the above expression for f makes sense. This is often considered the "natural domain", or the "maximal domain" of f.

[1] In general, if you don't assume that K is connected, you can show that f is constant on every connected component of K

[2] I prefer the term "target space" in place of co-domain.

(c) Compute $\lim_{(x,y)\to(1,-1)} f(x,y)$. You do not need a formal $\epsilon - \delta$ proof if you cite the correct theorems.

6. Consider the function

$$f(x,y) = \left(\frac{y\sin(x)}{x}, \tan\left(\frac{x}{y}\right), x^2 + y^2 - xy\right),$$

whose domain is yet to be determined.

(a) Identify the co-domain of f.

(b) As in the previous problem, find the natural domain of f.

(c) Compute $\lim_{(x,y)\to(1,-1)} f(x,y)$. Again, there is no need for a formal $\epsilon - \delta$ proof.

7. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, where $\mathbf{a} \in \mathbb{R}^n$, and $L \in \mathbb{R}^m$. Prove that there exists a constant M and an open set V with $\mathbf{a} \in V$ such that $\|f(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in V$.