Math 421	Homework 5	**Assigned**:	Monday, Februrary 24
Spring 2014	"Bolzano–Weierstrass and Cardinality"	**Due**:	Friday, February 28

Directions:

- You are still expected to work out all of the exercises.
- Volunteers will be asked to present solutions in class.
- Each solution you present will count towards your final homework grade.

HOMEWORK EXERCISES

1. Textbook exercises:

Section	Problem
9.1	9.1.7, 9.1.8
9.2	9.2.2

2. (a) Show that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is at most countable.

   (b) Show that the n-fold Cartesian product

$$\mathbb{N}^n := \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{n-\text{times}} = \Big\{(z_1, z_2, z_3, \ldots, z_n) : z_i \in \mathbb{N}, 1 \leq i \leq n\Big\}.$$

   is at most countable. Conclude that $\mathbb{Q}^n$ is a countable set.

3. Consider the set of all "words" taken from the alphabet $A = \{0, 1, 2, \ldots, 26\}$ given by

$$W := \Big\{(z_1, z_2, z_3, \ldots) : z_i \in A, \text{ and for some N}, z_k = 0 \text{ for all } k \geq N\Big\}.$$

   (a) Show that W is at most countable.

   (b) True or False. If we allow for words of infinite length, is

$$A^\omega := \Big\{(z_1, z_2, z_3, \ldots) : z_i \in A, i \geq 1\Big\} \tag{1}$$

   countable? For this problem, there is no need to provide a formal justification. *Hint:* Look at the next problem.

4. Consider the set of all infinite sequences consisting of zeros and ones

$$\{0, 1\}^\omega := \Big\{(z_1, z_2, z_3, \ldots) : z_i \in \{0, 1\}, i \in \mathbb{Z}_{\geq 1}\Big\}.$$

   Prove that $\{0, 1\}^\omega$ is uncountable. *Hint:* Consider base-2 expansions of numbers in $[0, 1]$, and construct an injective function from the $[0, 1]$ to $\{0, 1\}^\omega$.