Math 421	Homework 4	**Assigned**:	Monday, Februrary 10
Spring 2014	"Topology on $\mathbb{R}^n$"	**Due**:	Monday, February 17

Directions:

- All problems should be collocated, stapled and contain a single cover sheet with your name and a list of problems.
- Clearly mark one problem that you would like to have graded. Additional problem(s) will be selected randomly. Note that each exercise in the textbook counts as one problem!
- For each problem, submit only the final version of your solution. There should be enough details describing how you arrived at your solution, but do not submit "scratch-work" containing unnecessary computations.
- Homework is due at the *start of class* on the due date.

WARMUP PROBLEMS (Not to be turned in)

1. Textbook problems:

Section	Problem
8.4	8.4.1, 8.4.2

2. Prove one of the following identities concerning unions and intersections of sets:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. Given two sets, A, and B, prove one of the following identities:
   (a) $(A \cup B)^c = A^c \cap B^c$
   (b) $(A \cap B)^c = A^c \cup B^c$

   Note: These identities hold for arbitrary unions and intersections.

HOMEWORK EXERCISES

1. Textbook exercises:

Section	Problem
8.3	8.3.7
8.4	8.4.3, 8.4.6

2. Suppose $E \subseteq \mathbb{R}$ and $a, b \in E$, with $a < b$.

   (a) Recall that $[a, b] \subseteq E$ if and only if $\forall x \in [a, b]$, $x \in E$. The negation of this "if and only if" statement is given by:

   $$[a, b] \nsubseteq E \text{ if and only if } \neg\,(\forall x \in [a, b],\ x \in E).$$

   Expand the right hand side of the new (logically equivalent) statement.

   (b) If E is connected, show that $[a, b] \subseteq E$.

3. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of non-empty sets.

   (a) Determine the truth of each of the following statements:

      i. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $\exists \alpha \in \mathcal{A}$ such that $x \in U_\alpha$.
      ii. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $\forall \alpha \in \mathcal{A}$, $x \in U_\alpha$.
      iii. If $x \in \bigcap_{\alpha \in A} U_\alpha$, then $\exists \alpha \in \mathcal{A}$ such that $x \in U_\alpha$.
      iv. If $x \in \bigcap_{\alpha \in A} U_\alpha$, then $\forall \alpha \in \mathcal{A}$, $x \in U_\alpha$.

   For each case that is false, present a counterexample.

   (b) Write the converse of each statement in (a), and repeat.

4. If A is non-empty subset of $\mathbb{R}^n$, we define the *distance* from a point $\mathbf{x} \in \mathbb{R}^n$ and A as

   $$d(\mathbf{x}, A) := \inf_{\mathbf{a} \in A} \{\|\mathbf{x} - \mathbf{a}\|\}.$$

   (a) Show that if $\mathbf{x} \in A$, then $d(\mathbf{x}, A) = 0$. In fact, a stronger result holds:

   (b) Show that $\mathbf{x} \in \bar{A}$ if and only if $d(\mathbf{x}, A) = 0$.