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| **Math 421** | Homework 10 | **Assigned**: | Tuesday, April 22nd |
| Spring 2014 | "Differentiation" | **Due**: | Friday, April 25th |

**Directions:**

- Volunteers will be asked to present solutions in class.

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**WARMUP PROBLEMS** (Not to be turned in)

1. Write down the definition of a partial derivative.

2. Write down the definition of the (total) derivative. Compare with your answer to part 1.

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**HOMEWORK EXERCISES**

1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *independent of the second variable* if for each $x \in \mathbb{R}$, we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Show that $f$ is independent of the second variable if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$. What is $\frac{\partial f}{\partial x}$ in terms of $g$?

2. If $f : \mathbb{R}^2 \to \mathbb{R}$, and $\frac{\partial f}{\partial x_2} = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, show $f(x_1, x_2) = g(x_1)$ for some $g$. That is, show that $f$ is independent of the second variable. If $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$, show that $f$ is constant.

3. For each of the following functions $f$, find the matrix representation of a linear transformation $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - T(h)\|}{h} = 0.$$

   (a) $f(x) = (x^2, \sin(x))^T$.

   (b) $f(x) = \left(e^x, x^{1/3}, 1 - x^2\right)^T$.

4. Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and recall the definition of the *operator norm*,

$$\|T\| := \sup_{\mathbf{x} \neq 0} \frac{\|T(\mathbf{x})\|}{\|\mathbf{x}\|}.$$

   (a) Show that the supremum need only be taken over the unit sphere. That is, prove that $\sup_{\|\mathbf{x}\|=1} \|T(\mathbf{x})\| = \|T\|$.

   (b) Define

$$m := \inf \{C > 0 : \|T(\mathbf{x})\| \leq C\|\mathbf{x}\| \quad \text{for all } \mathbf{x}\}.$$

   Prove that $m = \|T\|$.

5. Suppose $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a linear function. Prove that $DT(\mathbf{a}) = T$ for all $\mathbf{a} \in \mathbb{R}^n$.

6. If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function, we define the *directional derivative* as

$$D_{\mathbf{v}} f(\mathbf{a}) := \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t},$$

   provided this limit exists.

   (a) Show that $D_{\mathbf{e}_i} f(\mathbf{a}) = \frac{\partial f}{\partial x_i}(\mathbf{a})$, provided both limits exist. Conclude that directional derivatives extend the definitions of partial derivatives.

   (b) If $c \in \mathbb{R}$ is a scalar, show that $D_{c\mathbf{v}} f(\mathbf{a}) = c D_{\mathbf{v}} f(\mathbf{a})$, provided the directional derivative exists.

   (c) If $f$ is differentiable at $\mathbf{a}$, show that $D_{\mathbf{v}} f(\mathbf{a}) = Df(\mathbf{a})\mathbf{v}$, where the left hand side denotes the directional derivative, and the right hand side denotes the multiplication of the total derivative against $\mathbf{v}$.

7. Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are considered to be *equal up to nth-order at $a$* if

$$\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h^n} = 0.$$

   (a) Show that $f$ is differentable at $a$ if and only if there is a function $g$ of the form $g(x) = a_0 + a_1(x - a)$ such that $f$ and $g$ are equal up to first order at $x = a$.

   (b) If $f'(a), f''(a), \ldots f^{(n)}(a)$ exist, show that $f$ and the function $g$ defined by

$$g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

   are equal up to $nth$-order at $a$. *Hint:* re-write the limit as $\lim_{x \to a}$ and use L'Hôspital's rule.

8. Suppose $f : V \to \mathbb{R}$ is a function defined on an open set $V$ containing the origin. Suppose further that $|f(\mathbf{x})| \leq \|\mathbf{x}\|^\alpha$, where $\alpha > 1$ is a scalar. Prove that $f$ is differentiable at the origin. What happens when you assume $\alpha = 1$?