Math 421	Homework 1	**Assigned**:	Sunday, January 12
Spring 2014	"Riemann Integral"	**Due**:	Wednesday, January 22

Directions:

- All problems should be collocated, stapled and contain a single cover sheet with your name and a list of problems.
- Clearly mark one problem which you would like to have graded. Additional problem(s) will be selected randomly.
- For each problem, submit only the final version of your solution. There should be enough details describing how you arrived at your solution, but do not submit "scratch-work" containing unnecessary computations.

WARMUP PROBLEMS (Not to be turned in)

1. If $E \subset \mathbb{R}$ has a finite infimum, and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that $\inf(E) \le a < \inf(E) + \epsilon$. Likewise, if E has a finite supremum, show that there exists a $b \in E$ such that $\sup(E) \ge b > \sup(E) - \epsilon$.

2. If $A, B \subseteq \mathbb{R}$, prove that $\sup(A \cup B) \ge \sup(A)$ and that $\inf(A \cup B) \le \inf(A)$.

3. Consider the function
$$f(x) = \begin{cases} 0, & x \le 1/2 \\ 1, & 1/2 < x \le 1 \\ 0, & x > 1. \end{cases}$$
Prove that f is (Riemann) integrable on $[0, 2]$.

HOMEWORK EXERCISES

1. Define the set $E = \{1/n : n \in \mathbb{N}\} \subseteq [0, 1]$. Prove that the (characteristic) function
$$f(x) = \begin{cases} 1, & x \in E \\ 0, & \text{otherwise}, \end{cases}$$
is (Riemann) integrable on $[0, 1]$. What is the value $\int_0^1 f(x)\,dx$?

2. Consider the *floor* function defined by
$$f(x) = \lfloor x \rfloor = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ & \vdots \end{cases}$$

If $n \in \mathbb{Z}_{\geq 1}$, prove that f is (Riemann) integrable on $[0, n]$.

3. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$, and a partition $P = \{x_0, x_1, \ldots x_n\}$ of $[0,1]$.

   (a) Compute $m_i = \inf_{t\in[x_{i-1},x_i]} f(t)$ and $M_i = \sup_{t\in[x_{i-1},x_i]} f(t)$.

   (b) Compute the lower $L(f, P)$ and upper $U(f, P)$ Riemann sums of f with respect to the partition P.

   (c) Let $\epsilon > 0$, and suppose P satisfies $\|P\| < \epsilon/2$. Show that $U(f, P) - L(f, P) < \epsilon$, and conclude that f is integrable. *Hint:* find an upper bound for $M_i - m_i$, and then sum over $(M_i - m_i)\Delta x_i$.

4. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/q & x = p/q \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

   In order for f to be well-defined, We assume that $p, q \in \mathbb{Z}$ have no common factors. For example, $f(4/6) = f(2/3) = 1/3$. Prove that f is (Riemann) integrable.

5. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.

   (a) If f is continuous at some point $x_0 \in [a, b]$, and $f(x_0) \neq 0$, prove that

$$(L) \int_a^b |f(x)|\, dx > 0.$$

   (b) If f is continuous on $[a, b]$, prove that $\int_a^b |f(x)|\, dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

   (c) Does part (b) hold if the absolute values are removed? Prove or provide a counterexample.