# NOTES ON LECTURE 002: HW #2

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**1–7, #16** *Prove that if $mn$ is even, then $m$ is even or $n$ is even.*

The contrapositive of this theorem is "If $\neg$($m$ is even or $n$ is even ), then $\neg$($mn$ is even ). The negations of both of these become: "If $m$ and $n$ is odd, then $mn$ is odd." So, let's prove the contrapositive.

*Proof:* Suppose $m$ and $n$ are odd. Then there exists integers, $k, l \in \mathbb{Z}$ with $m = 2k + 1$ and $n = 2l + 1$. The product is then,

$$mn = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2\,(2kl + k + l) + 1,$$

which is an odd number. □

Some common things that we, as a class can improve on are the following:

(1) If you introduce something new, make sure you tell the reader what you are doing, and tell the reader why you're introducing the new object. For example, the following statement introduces two variables, doesn't tell the reader where they came from, or why they are being introduced. The whole statement is left wanting.
    "Suppose $m = 2k + 1$ and $n = 2l + 1$. Their product is $mn = \ldots$"
(2) Use full sentences to convey full thoughts. The following argument is incomplete:
    "$m = 2k + 1, n = 2l + 1$. $mn = \ldots$"
(3) The integer $k$ for m and the integer $l$ for n are possibly different! Therefore, the following statement is simply wrong:
    "Suppose $m$ and $n$ are odd. That is $m = 2k+1$ and $n = 2k+1$ for some $k \in \mathbb{Z}$."
(4) A proof by cases is not the way to prove this problem. Yes, you can assume that there are 4 cases for what $m$ and $n$ can possibly be, one of which must be ruled out in order to satisfy the hypotheses, and the remaining three satisfy the conclusion, but this is way too messy. Just prove the contrapositive, since this is one of the cases you must rule out! Speaking of which,
(5) You do not need a contradiction to prove this problem. The contrapositive of $(p \to q)$ is $(\neg q \to \neg p)$. These are logically identical!

A proof by contradiction means you assume $P(x)$ and $\neg Q(x)$, and then take steps and run into a contradiction. While a proof by contraposition may feel like a contradiction, they are very different. You can think of a proof by contraposition as a direct proof in the following sense. First, you assume $\neg Q(x)$, then take steps to show $\neg P(x)$.

A proof by contradiction means you touch both the hypothesis, $P(x)$ as well as the conclusion, $\neg Q(x)$, and show that a contradiction always arises. The reason this works to prove the statement is the following. We know that $p \to q \equiv \neg p \vee q$, and if you can show that $p \wedge \neg q \equiv F$, then you're done, since then we would have

$$T \equiv \neg F \equiv \neg (p \wedge \neg q) \equiv \neg p \vee q \equiv p \to q.$$

Note: I'm cheating a little bit here. I should have written $P(x)$ in place of $p$, and $Q(x)$ in place of $q$. But, imagine fixing an $x$, in which case $P(x) = p$ can only take on a true or false value.