- 1. Describe the elements of the set $(\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}$. Is this set countable or uncountable?
- 2. Let $A = \{\emptyset, \{\emptyset\}\}$. What is the cardinality of A? Is $\emptyset \subset A$? Is $\emptyset \in A$? Is $\{\emptyset\} \subset A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset\} \in A$?
- 3. List the elements of the set $A \times B$ where A is the set in the previous question and $B = \{1, 2\}$.
- 4. Suppose that A, B, and C are sets. Which of the following statements is true for all sets A, B, and C? For each, either prove the statement or give a counterexample: $(A \cap B) \cup C = A \cap (B \cup C)$, $A \cap B \subseteq A \cup B$, if $A \subset B$ then $A \times A \subset A \times B$, $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.
- 5. State the negation of each of the following statements:
 - There exists a natural number m such that $m^3 m$ is not divisible by 3.
 - $\sqrt{3}$ is a rational number.
 - 1 is a negative integer.
 - 57 is a prime number.
- 6. Verify the following laws:
 - (a) Let P, Q and R are statements. Then,

 $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.

• (b) Let P and Q are statements. Then,

 $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.

- 7. Write the open statement P(x, y): "for all real xand y the value $(x - 1)^2 + (y - 3)^2$ is positive" using quantifiers. Is the quantified statement true or false? Explain.
- 8. Prove that 3x + 7 is odd if and only if x is even.
- 9. Prove that if a and b are positive numbers, the $\sqrt{ab} \leq \frac{a+b}{2}$. This is referred to as "Inequality between geometric and arithmetic mean."
- 10. Let A, B, and C be sets. Prove that $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C).$
- 11. Let A, B, and C be sets. Prove that $(A B) \bigcap (A C) = A (B \bigcup C).$
- 12. Suppose that x and y are real numbers. Prove that if x + y is irrational, then x is irrational or y is irrational.

- 13. Let x be an irrational number. Prove that x^4 or x^5 is irrational.
- 14. Use a proof by contradiction to prove the following.

There exist no natural numbers m such that $m^2 + m + 3$ is divisible by 4.

- 15. Let a, b be distinct primes. Then $\log_a(b)$ is irrational.
- 16. Prove or disprove the statement: there exists an integer n such that $n^2 3 = 2n$.
- 17. Prove or disprove the statement: there exists a real number x such that $x^4 + 2 = 2x^2$.
- 18. Prove that there exists a unique real number x such that $x^3 + 2 = 2x$.
- 19. Disprove that statement: There exists integers a and b such that $a^2 + b^2 \equiv 3 \pmod{4}$
- 20. Use induction to prove that $6|(n^3 + 5n)$ for all $n \ge 0$.
- 21. Use induction to prove that

$$1 \cdot 4 + 2 \cdot 7 + \dots + n(3n+1) = n(n+1)^2$$

for all $n \in \mathbb{N}$.

- 22. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \ge 11$, there are nonnegative integers x and y such that n = 4x + 5y.
- 23. A sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = -2$ and for $n \ge 1$,

$$a_{n+1} = 5a_n - 6a_{n-1}.$$

Prove that for $n \ge 0$,

 $a_n = 5 \times 2^n - 4 \times 3^n.$

- 24. Suppose R is an equivalence relation on a set A. Prove or disprove that R^{-1} is an equivalence relation on A.
- 25. Consider the set $A = \{a, b, c, d\}$, and suppose R is an equivalence relation on A. If R contains the elements (a, b) and (b, d), what other elements must it contain?
- 26. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Find a relation on $A \times B$ that is transitive and symmetric, but not reflexive.
- 27. Suppose A is a finite set and R is an equivalence relation on A.

- (a) Prove that $|A| \leq |R|$.
- (b) If |A| = |R|, what can you conclude about R?
- 28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

 $R = \{ (x \mod 4, 3x \mod 6) \mid x \in \mathbb{Z} \}.$

Prove that R is a function from \mathbb{Z}_4 to \mathbb{Z}_6 . Is R a bijective function?

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

 $S = \{ (x \mod 4, 2x \mod 6) \mid x \in \mathbb{Z} \}.$

Prove that S is not a function from \mathbb{Z}_4 to \mathbb{Z}_6 .

- 30. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by h(a, x) = (f(a), g(x)). Prove that h is bijective.
- 31. Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.
- 32. (X points) Let \mathbb{R}^+ denote the set of positive real numbers and let A and B be denumerable subsets of \mathbb{R}^+ . Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.
- 33. Prove that the interval (0, 1) is numerically equivalent to the interval $(0, +\infty)$.
- 34. Prove the following statement: A nonempty set S is **countable** if and only if there exists an injective function $g: S \to \mathbb{N}$.
- 35. Consider the set $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Prove that $\mathbb{R} S$ is uncountable.
- 36. (a) Suppose A, B are sets. Prove that if A and B have the same cardinality, then $A \times \mathbb{Z}$ and $B \times \mathbb{Z}$ have the same cardinality.

(b) Prove that \mathbb{Z}^n has the same cardinality as \mathbb{Z}^{n+1} for all $n \in \mathbb{N}$. *Hint: Induct on n, and use part (a) for the inductive step.*

(c) Prove that \mathbb{Z}^n is countable for all $n \in \mathbb{N}$.

37. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.

- 38. Let $a, b, c \in \mathbb{Z}$. Prove that if c is a common divisor of a and b, then c divides any linear combination of a and b.
- 39. Define the term "p is a prime". Then prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a, then gcd(a, p) = 1.
- 40. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by gcd(a, b, c). Assume that a and b are not both zero. Prove the following equation:

$$gcd(a, b, c) = gcd(gcd(a, b), c).$$

41. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n \to \infty} \frac{3n+1}{n-2} = 3.$$

42. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n \to \infty} \frac{(-1)^n 3n + 1}{n - 2}$$

does not exist.

43. Let (a_n) be a sequence with positive terms such that $\lim_{n\to\infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n \to \infty} \frac{3a_n + 1}{2} = 2.$$

44. (a) Use induction to prove

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{2n(2n+2)} = \frac{n}{4(n+1)}$$

for all $n \in \mathbb{N}$.

(b) Prove
$$\sum_{k=1}^{\infty} \frac{1}{2k(2k+2)} = \frac{1}{4}$$