1. Describe the elements of the set $(\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}$. Is this set countable or uncountable?

   Solution: The set is equal to $\{(x, y) \mid x \in \mathbb{Z}, y \in \mathbb{N}\} = \mathbb{Z} \times \mathbb{N}$. Since the Cartesian product of two denumerable sets is denumerable, this set is denumerable, hence countable.

2. Let $A = \{\emptyset, \{\emptyset\}\}$. What is the cardinality of A? Is $\emptyset \subset A$? Is $\emptyset \in A$? Is $\{\emptyset\} \subset A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset, \{\emptyset\}\} \in A$?

   Solution: $|A| = 2$; it has two elements: $\emptyset$ and $\{\emptyset\}$. The answers to the remaining questions are yes, yes, yes, yes, no.

3. List the elements of the set $A \times B$ where A is the set in the previous question and $B = \{1, 2\}$.

   Solution: $A \times B = \{(\emptyset, 1), (\emptyset, 2), (\{\emptyset\}, 1), (\{\emptyset\}, 2)\}$.

4. Suppose that A, B, and C are sets. Which of the following statements is true for all sets A, B, and C? For each, either prove the statement or give a counterexample: $(A \cap B) \cup C = A \cap (B \cup C)$, $A \cap B \subseteq A \cup B$, if $A \subset B$ then $A \times A \subset A \times B$, $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.

   Solution: $(A \cap B) \cup C \neq A \cap (B \cup C)$ in general; a counterexample is $A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}$. Then $(A \cap B) \cup C = \{1, 4\}$, whereas $A \cap (B \cup C) = \{1\}$.

   $A \cap B \subset A \cup B$ is true. If $x \in A \cap B$, then $x \in A$. So, $x \in A \cup B$.

   $A \subset B \implies A \times A \subset A \times B$ is true. If $(x, y) \in A \times A$, then $x, y \in A$. Therefore, $y \in B$. Therefore, $(x, y) \in A \times B$.

   $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$ is true. Recall that $\cap$ and $\cup$ satisfy associative laws. Thus,

   $$\overline{A} \cap \overline{B} \cap \overline{C} = (\overline{A} \cap \overline{B}) \cap \overline{C} = \overline{A \cup B} \cap \overline{C},$$

   by De Morgan's law. Another application of De Morgan's law yields

   $$\overline{(A \cup B) \cup C} = \overline{A \cup B \cup C}.$$

5. State the negation of each of the following statements:

   - There exists a natural number m such that $m^3 - m$ is not divisible by 3.
   - $\sqrt{3}$ is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:

   - (a) Let P, Q and R are statements. Then,

     $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.

   - (b) Let P and Q are statements. Then,

     $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.

7. Write the open statement $P(x, y)$: **"for all real x and y the value $(x-1)^2 + (y-3)^2$ is positive"** using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that $3x + 7$ is odd if and only if x is even.

   Solution: First, we will prove that if x is even, then $3x + 7$ is odd. Assume x is even. Then $\exists k \in \mathbb{Z}$ such that $x = 2k$. Therefore, $3x + 7 = 6k + 7 = 2(3k + 3) + 1 = 2s + 1$, where $s = 3k + 3 \in \mathbb{Z}$. Thus, $3x + 7$ is odd. Now, we need to prove that if $3x + 7$ is odd, then x is even. We are going to prove the equivalent, contrapositive statement. Assume x is odd. Then $\exists k \in \mathbb{Z}$ such that $x = 2k + 1$. Therefore, $3x + 7 = 6k + 3 + 7 = 2(3k + 5) = 2s$, where $s = 3k + 5 \in \mathbb{Z}$. Thus, $3x + 7$ is even. Thus, $3x + 7$ is odd if and only if x is even.

9. Prove that if a and b are positive numbers, the $\sqrt{ab} \leq \frac{a+b}{2}$. This is referred to as "Inequality between geometric and arithmetic mean."

   Solution: Let $a, b \in \mathbb{R}^+$. Then $(a - b) \in \mathbb{R}$ and thus $(a-b)^{\geq}0$. The following inequalities are equivalent.

   $$\begin{aligned}(a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ a^2 + 2ab + b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab \\ a + b &\geq 2\sqrt{ab} \\ \frac{a+b}{2} &\geq \sqrt{ab}.\end{aligned}$$

   Thus, we have arrived at the desired inequality, which holds true for all $a, b \in \mathbb{R}$.

10. Let A, B, and C be sets. Prove that $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$.

    Solution: First, we will prove that $A \times (B \bigcap C) \subseteq (A \times B) \bigcap (A \times C)$. Let $(x, y) \in A \times (B \bigcap C)$ be an arbitrary element. Then, $x \in A$ and $y \in B$ and $y \in C$. Thus, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Therefore, $(x, y) \in (A \times B) \bigcap (A \times C)$. Thus, we can conclude that $A \times (B \bigcap C) \subseteq (A \times B) \bigcap (A \times C)$.

Now, we need to prove that that $(A \times B)\bigcap(A \times C) \subseteq A \times (B \bigcap C)$. Take an arbitrary element $(x, y) \in (A \times B)\bigcap(A \times C)$. Then, $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$. Therefore, $x \in A$ and $y \in B$ and $y \in C$. Thus, $y \in B \bigcap C$, which implies $(x, y) \in A \times (B \bigcap C)$.

Since we have proven both inclusions, we can conclude the desired equality of sets, namely, $A \times (B \bigcap C) = (A \times B)\bigcap(A \times C)$.

11. Let A, B, and C be sets. Prove that $(A - B)\bigcap(A - C) = A - (B \bigcup C)$.

12. Suppose that x and y are real numbers. Prove that if $x + y$ is irrational, then x is irrational or y is irrational.

    Solution: We will instead prove the contrapositive statement, which is equivalent to the original one. Assume that $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Then $\exists p, q, r, s \in \mathbb{Z}$ such that $x = \frac{p}{q}$ and $y = \frac{r}{s}$. Then $x + y = \dfrac{sp + qr}{sq} \in \mathbb{Z}$. (Alternatively, we can use the fact that Q is closed under addition.) Thus, if x and $y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$.

13. Let x be an irrational number. Prove that x^4 or x^5 is irrational.

    Solution: We will instead prove the contrapositive statement, which is equivalent to the original one, namely, if x^4 and x^5 are rational, then x is rational. Clearly, if $x^5 = 0$, then $x = 0$, thus this case is trivial. Thus, assume that x^5 and $x^4 \in \mathbb{Q} - \{0\}$. Then $\exists p, q, r, s \in \mathbb{Z} - \{0\}$ such that $x^5 = \frac{p}{q}$ and $x^4 = \frac{r}{s}$. Thus, $x = \dfrac{x^5}{x^4} = \frac{ps}{qr} \in \mathbb{Q}$. This concludes the proof of the contrapositive statement, thus the original statement also holds true.

14. Use a proof by contradiction to prove the following.

    There exist no natural numbers m such that $m^2 + m + 3$ is divisible by 4.

    Hint: Consider two cases: n is even, and n is odd.

15. Let a, b be distinct primes. Then $\log_a(b)$ is irrational.

16. Prove or disprove the statement: there exists an integer n such that $n^2 - 3 = 2n$.

17. Prove or disprove the statement: there exists a real number x such that $x^4 + 2 = 2x^2$.

18. Prove that there exists a unique real number x such that $x^3 + 2 = 2x$.

19. Disprove that statement: There exists integers a and b such that $a^2 + b^2 \equiv 3 \pmod 4$

20. Use induction to prove that $6|(n^3 + 5n)$ for all $n \geq 0$.

21. Use induction to prove that

    $$1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2$$

    for all $n \in \mathbb{N}$.

22. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 11$, there are nonnegative integers x and y such that $n = 4x + 5y$.

23. A sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = -2$ and for $n \geq 1$,

    $$a_{n+1} = 5a_n - 6a_{n-1}.$$

    Prove that for $n \geq 0$,

    $$a_n = 5 \times 2^n - 4 \times 3^n.$$

    Solution: Since $a_0 = 5 \times 2^0 - 4 \times 3^0 = 5 - 4 = 1$, the formula holds for $n = 0$.

    Suppose for some integer $k \geq 0$, $a_i = 5 \times 2^i - 4 \times 3^i$ for all integers i with $0 \leq i \leq k$.

    If $k = 0$, then

    $$a_{k+1} = a_1 = 5 \times 2^1 - 4 \times 3^1 = 10 - 12 = -2.$$

    So the formula holds for $k + 1$.

    Now we assume $k \geq 1$. Since $k + 1 \geq 2$, $k, k - 1 \geq 0$. Hence,

    $$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} \\ &=5(5 \times 2^k - 4 \times 3^k) - 6(5 \times 2^{k-1} - 4 \times 3^{k-1}) \\ &=25 \times 2^k - 20 \times 3^k - 30 \times 2^{k-1} + 24 \times 3^{k-1} \\ &=25 \times 2^k - 20 \times 3^k - 15 \times 2^k + 8 \times 3^k \\ &=10 \times 2^k - 12 \times 3^k \\ &=5 \times 2^{k+1} - 4 \times 3^{k+1}. \end{aligned}$$

    So the formula also holds for $k + 1$.

    By the Strong Principle of Mathematical Induction, for every $n \geq 0$,

    $$a_n = 5 \times 2^n - 4 \times 3^n.$$

24. Suppose R is an equivalence relation on a set A. Prove or disprove that R^{-1} is an equivalence relation on A.

    Solution: If R is an equivalence relation, then so is $R^{-1} = \{(y, x) \in A \times A \mid (x, y) \in R\}$.

    Proof 1: Let $a \in A$. Then since R is reflexive we have $(a, a) \in R$. It follows from the definition of R^{-1} that $(a, a) \in R^{-1}$, proving that R^{-1} is reflexive as well. To show that R^{-1} is symmetric, let $(a, b) \in R^{-1}$. Then by definition $(b, a) \in R$. Since R is symmetric, $(a, b) \in R$ as well, and so $(b, a) \in R^{-1}$. To prove that R^{-1} is transitive, let $(a, b), (b, c) \in R^{-1}$. Then $(b, a), (c, b) \in R$, and since R is symmetric, it follows that $(a, b), (b, c) \in R$. By the transitivity of R, we have $(a, c) \in R$ and so $(c, a) \in R^{-1}$. Finally, since R^{-1} is symmetric, it follows that $(a, c) \in R^{-1}$, which shows R^{-1} is transitive.

    Proof 2: We will show that $R = R^{-1}$, and so R^{-1} will automatically be an equivalence relation because we have assumed R is. Let $(a, b) \in R$. Since R is symmetric, $(b, a) \in R$. By the definition of R^{-1} it follows that $(a, b) \in R^{-1}$, which shows $R \subseteq R^{-1}$. The reverse inclusion is similar.

25. Consider the set $A = \{a, b, c, d\}$, and suppose R is an equivalence relation on A. If R contains the elements (a, b) and (b, d), what other elements must it contain?

    Solution: In addition to (a, b) and (b, d), the equivalence relation R must contain

$$(a, a), (b, b), (c, c), (d, d)$$
$$(b, a), (d, b)$$
$$(a, d)$$
$$(d, a)$$

    The elements in the first row appear due to reflexivity; the elements in the second are due to symmetry; the element in the third row is due to transitivity; the element in the last row is due to symmetry from the previous row.

26. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Find a relation on $A \times B$ that is transitive and symmetric, but not reflexive.

    Solution 1: Take $R = \emptyset \subset (A \times B) \times (A \times B)$.

    Solution 2: Take $R = \{((a_1, b_1), (a_1, b_1))\}$. This is obviously symmetric (switch (a_1, b_1) with itself), and it is transitive. It is not reflexive because it is missing, say, $((a_2, b_1), (a_2, b_1))$.

    There are many other solutions that are possible. Note that if $((a_i, b_j), (a_k, b_l))$ is in the relation, then so is $((a_k, b_l), (a_i, b_j))$ by symmetry, and hence $((a_i, b_j), (a_i, b_j))$ and $((a_k, b_l), (a_k, b_l))$ are in the relation as well. In particular, to ensure that it is not reflexive, you need to make sure there is at least one element of $A \times B$ that does not appear as a component of any element of the relation.

27. Suppose A is a finite set and R is an equivalence relation on A.

    (a) Prove that $|A| \leq |R|$.
    Solution: Since R is reflexive, if $a \in A$ then $(a, a) \in R$. In particular, the map $f : A \to R$ defined by $f(a) = (a, a)$ is well-defined. This is obviously injective, and so $|A| \leq |R|$.

    (b) If $|A| = |R|$, what can you conclude about R?
    Solution: If $|A| = |R|$ then R contains no more elements than those in the image of f from part (a). This implies that $R = \{(a, a) \mid a \in A\}$ is the diagonal equivalence relation.

28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$R = \{(x \bmod 4, 3x \bmod 6) \mid x \in \mathbb{Z}\}.$$

    Prove that R is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. Is R a bijective function?

    Solution: We need to show two things: (1) For every $a \in \mathbb{Z}_4$ there is some $b \in \mathbb{Z}_6$ such that $(a, b) \in R$; (2) If $(a, b), (a, b') \in R$ then $b = b'$. The first follows immediately from the definition of R: if $a = [x] \in \mathbb{Z}_4$, and $x \in [x]$ is any integer, then take b to be the mod 6 reduction of x, and so we have $(a, b) \in \mathbb{Z}_4 \times \mathbb{Z}_6$. To prove (2), suppose $(a, b), (a, b') \in R$. Then we have

$$(a, b) = (x \bmod 4, 3x \bmod 6),$$

$$(a, b') = (y \bmod 4, 3y \bmod 6)$$

    for some integers x, y. We obviously have $x \bmod 4 = y \bmod 4$ and so $x = y + 4k$ for some integer k. This gives $3x = 3y + 12k$ and so $b = 3x \pmod 6 = 3y \pmod 6 = b'$, as desired.

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$S = \{(x \bmod 4, 2x \bmod 6) \mid x \in \mathbb{Z}\}.$$

    Prove that S is not a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

Solution: This fails item (2) in the solution to the previous problem (it satisfies item (1)): We have

$$0 \pmod 4 = 4 \pmod 4,$$

but

$$2 \cdot 0 \pmod 6 \neq 2 \cdot 4 \pmod 6.$$

30. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that h is bijective.

    Solution: First we show h is injective. Suppose $h(a, x) = h(a', x')$. Then $f(a) = f(a')$ and $g(x) = g(x')$. Since each of these is injective, it follows that $a = a'$ and $x = x'$, which is equivalent to saying $(a, x) = (a', x')$.

    To see that h is surjective, let $(b, y) \in B \times Y$. Then since f, g are surjective, there are $a \in A$ and $x \in X$ such that $f(a) = b$ and $g(x) = y$. It follows that $h(a, x) = (b, y)$.

31. Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.

    Solution: The direction ($\Leftarrow$) is false. Indeed, consider the case where $A = B$, and take f to be the identity function (this is obviously injective). Now take g to be any function that is surjective but not injective. Then $g \circ f = g$ is not injective, and so certainly not bijective.

    The direction ($\Rightarrow$) is true. To see this, suppose $g \circ f$ is bijective. If $f(a) = f(a')$, then $(g \circ f)(a) = (g \circ f)(a')$ and so $a = a'$ since $g \circ f$ is injective. To see surjectivity, let $c \in C$. Then since $g \circ f$ is surjective, it follows that there is some $a \in A$ with $(g \circ f)(a) = c$. Now take $b = f(a)$, and so $g(b) = c$.

32. *(X points)* Let $\mathbb{R}^+$ denote the set of positive real numbers and let A and B be denumerable subsets of $\mathbb{R}^+$. Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.

33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.

    Solution: The function $(0, 1) \to (0, \infty)$ defined by sending $x \in (0, 1)$ to $\tan(2x/\pi)$ is a bijection.

34. Prove the following statement: A nonempty set S is **countable** if and only if there exists an injective function $g : S \to \mathbb{N}$.

    Solution: First assume S is countable. Then S is either finite or there is a bijection $f : \mathrm{N} \to S$. We leave the case where S is finite to the reader. In the case where there is a bijection f, then the inverse of f is an injection from S to $\mathbb{N}$. Conversely, if there is an injection $g : S \to \mathbb{N}$, then S has the same cardinality as its image $g(S) \subset \mathbb{N}$. If the image is finite, then S is countable. If the image is infinite, then $g(S)$ is an infinite subset of a countable set and so is countable. In either case S is countable.

35. Consider the set $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Prove that $\mathbb{R} - S$ is uncountable.

    Solution: First observe that the set S is countable. Indeed, the function $F : \mathbb{Z} \times \mathbb{Z} \to S$ defined by $F(a, b) = a + b\sqrt{2}$ is a bijection (the reader should check this). Next, assume that $\mathbb{R} - S$ is countable. Then $\mathbb{R} = (\mathbb{R} - S) \cup S$ would be the union of two countable sets, and so would be countable. However, this is a contradiction since $\mathbb{R}$ is uncountable.

36. (a) Suppose A, B are sets. Prove that if A and B have the same cardinality, then $A \times \mathbb{Z}$ and $B \times \mathbb{Z}$ have the same cardinality.

    Solution: Since A, B have the same cardinality, there is some bijection $f : A \to B$. Define a function $F : A \times \mathbb{Z} \to B \times \mathbb{Z}$ by $F(a, n) = (f(a), n)$. Then F is a bijection (the reader should check this), so $A \times \mathbb{Z}$ and $B \times \mathbb{Z}$ have the same cardinality.

    (b) Prove that $\mathbb{Z}^n$ has the same cardinality as $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{N}$. *Hint: Induct on n, and use part (a) for the inductive step.*

    Solution: The base case is that $\mathbb{Z}$ has the same cardinality as $\mathbb{Z}^2$. This is basically Result 10.6 from the book. For the inductive step, use part (a) with $A = \mathbb{Z}^n$ and $B = \mathbb{Z}^{n+1}$.

    (c) Prove that $\mathbb{Z}^n$ is countable for all $n \in \mathbb{N}$.

    Solution: We know that $\mathbb{Z}$ is countable. Since the relation of 'countable' is transitive, part (c) follows from part (b).

37. Compute the greatest common divisor of 42 adn 13 and then express the greatest common divisor as a linear combination of 42 and 13.

    Solution: $42 = 39 + 3 = 3(13) + 3$; $13 = 12 + 1 = 4(3) + 1$; $3 = 3(1) + 0$. Therefore, the gcd is equal to 1. Working backwards, we have that $1 = 13 - 4(3) = 13 - 4(42 - 3(13) = 13(13) + (-4)42$.

38. Let $a, b, c \in \mathbb{Z}$. Prove that if c is a common divisor of a and b, then c divides any linear combination of a and b.

Solution: Suppose c is a common divisor of a and b and let $ax + by$, where $x, y \in \mathbb{Z}$, be a linear combination of a and b. Then $c \mid a$ and $c \mid b$. Therefore, $a = cm$ and $b = cn$ for some $m, n \in \mathbb{Z}$. It follows that $ax + by = cmx + cny = c(mx + ny)$. Therefore, $c \mid (ax + by)$.

39. Define the term "p is a prime". Then prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a, then $\gcd(a, p) = 1$.

Solution: A number p is prime if p is a positive integer greater than one and whenever $p = ab$ for some positive integers a and b, then $a = 1$ or $b = 1$.

Suppose that p is prime and that $a \in \mathbb{Z}$ is not divisible by p. Since p and a are not both zero, there is a greatest common divisor d. If $d > 1$, then $d \mid p$ implies that $d = p$ since the only divisors of p are 1 and p. Since $d \mid a$, this implies that $p \mid a$ which is a contradiction. Therefore, d cannot be greater than 1. Hence, $d = 1$.

40. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by $\gcd(a, b, c)$. Assume that a and b are not both zero. Prove the following equation:

$$\gcd(a, b, c) = \gcd(\gcd(a, b), c).$$

Solution: Let d be the gcd of a, b, and c. Let e be the gcd of a and b. Let f be the gcd of e and c. We prove that $d = f$. Since e is a linear combination of a and b, $d \mid e$. Since $d \mid c$, and f is a linear combination of e and c, it follows that d divides f. Therefore $d \leq f$.

On the other hand, $f \mid e$ and $f \mid c$. Since $e \mid a$ and $e \mid b$, $f \mid a$ and $f \mid b$. Thus, f is a common divisor of a, b, and c. Hence, $f \leq d$. Therefore, $f = d$.

41. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n\to\infty} \frac{3n+1}{n-2} = 3.$$

42. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n\to\infty} \frac{(-1)^n 3n+1}{n-2}$$

does not exist.

43. Let (a_n) be a sequence with positive terms such that $\lim_{n\to\infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n\to\infty} \frac{3a_n+1}{2} = 2.$$

44. (a) Use induction to prove

$$\frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \cdots + \frac{1}{2n(2n+2)} = \frac{n}{4(n+1)}$$

for all $n \in \mathbb{N}$.

(b) Prove $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k+2)} = \frac{1}{4}$.