Math 451	Homework 6	**Assigned**:	Wednesday, November 18
Fall 2013	"Numerical Integration"	**Due**:	Friday, December 6

Directions:

- Include a cover page.
- For each problem, submit only the final version of your solution. Each problem should be collocated and stapled.
- Always label every plot (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB (or matplotlib) when comparing 2 or more plots to make comparison easier and to save paper.

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1. Consider the following 4 equally spaced points on the interval $[a, b]$:

$$x_j = a + jh, \qquad j = 0, 1, 2, 3\,,$$

where $h = (b - a)/3$.

(a) By hand, construct all the Lagrange polynomials $L_{3,j}(x)$ that correspond to the points x_0, x_1, x_2, and x_3.

(b) Use these Lagrange polynomials to construct the interpolating polynomial $P_3(x)$ that interpolates the function $f(x)$ at the points x_0, x_1, x_2, and x_3.

(c) Integrate the interpolating polynomial $P_3(x)$ to derive the following closed Newton-Cotes formula, often referred to as **Simpson's three-eigths rule**:

$$\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}\Big[f(x_0) + 3f(x_0+h) + 3f(x_0+2h) + f(x_0+3h)\Big]\,.$$

2. **SOURCE CODE:**

Given an interval $[a, b]$, consider the following $n + 1$ equally spaced points:

$$x_j = a + jh, \qquad j = 0, 1, 2, \ldots, n\,,$$

where $h = (x_n - x_0)/n$. Decompose the integral of $f(x)$ into $n/3$ equally spaced portions:

$$\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=1}^{n/3} \int_{x_{3i-3}}^{x_{3i}} f(x)\,dx.$$

On each subinterval $[x_{3i-3}, x_{3i}]$ there are 4 equally spaced data points.

(a) Derive the the **Composite Simpson's three-eigths rule**, by applying the 3-8ths rule to each subinterval $[x_{3i-3}, x_{3i}]$:

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i=1}^{n/3}\Big\{f(x_{3i-2}) + f(x_{3i-1})\Big\} + 2\sum_{i=1}^{n/3-1} f(x_{3i}) + f(x_n)\right].$$

(b) Write the following routine in MATLAB:

- `I = CompSimp38(@f, a, b, n)`

where `a` and `b` are the left and right endpoints of the integral and `n` is the number points used in the integration (**NOTE:** `n` must be divisible by 3!), and `f` is a function handle.

3. **APPLICATION:**

Consider the following integral:

$$I = \int_3^4 \frac{x}{\sqrt{x^2-4}}\,dx.$$

(a) Compute this integral exactly by hand. Show all of your steps.

(b) Use your `CompSimp38` function to find a convergence rate.

(c) What happen's if you use `CompSimp38` to evaluate $I = \int_2^4 \frac{x}{\sqrt{x^2-4}}\,dx$?

4. An integral over an infinite, or semi-infinite domain is referred to as an **improper integral**, and leads to complications for numerical approximations. Using your composite Simpson's rule, evaluate the integral

$$I = \int_0^\infty \frac{e^{-x}}{1+x^2}dx = 0.621449624235813\ldots$$

using $n = 2, 4, 8, \ldots 128$, and with each of the three following approaches:

(i) Truncation of limits: Since $e^{-36} \approx \epsilon$ for double precision, apply your quadrature rule to:

$$I \approx \int_0^{36} \frac{e^{-x}}{1+x^2}dx.$$

(ii) Transformation: let

$$x = \tan y, \quad dx = \sec^2 y dy, \quad \Longrightarrow I = \int_0^{\frac{\pi}{2}} e^{-\tan y} dy.$$

(iii) Transformation: let

$$x = \frac{1-y}{1+y}, \quad dx = \frac{-2dy}{(1+y)^2}, \quad \Longrightarrow I = \int_{-1}^{1} \frac{e^{-\left(\frac{1-y}{1+y}\right)}}{1+y^2}dy.$$

Which of these approaches gives the best result? What can you conclude about integrals that have infinite domains?