Math 451	Homework 5	**Assigned**:	Wednesday, November 18
Fall 2013	"Interpolation: Part II"	**Due**:	Friday, December 6

Directions:

- Include a cover page.
- For each problem, submit only the final version of your solution. Each problem should be collocated and stapled.
- Always label every plot (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB (or matplotlib) when comparing 2 or more plots to make comparison easier and to save paper.

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1. Why are "boundary conditions" required to be defined when constructing cubic spline interpolants? (e.g. What happens if we don't define them?)

2. Consider the following definition:

   Definition. *The* Natural boundary conditions, *or free boundary conditions for a cubic spline are defined by forcing* $s''(a) = s''(b) = 0$. *Because* $s''(a) = c_0$ *and* $s''(b) = c_n$*, these translate to* $c_0 = c_n = 0$.

   For a uniform partition:

   $$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b,$$

   write down the matrix for the unknown coefficients c_j that defines the system with natural boundary conditions. *Hint:* your system should be of the form: $Ac = a$, where $c = (c_0, c_1, ..., c_5)^T$ and $a = (a_0, a_1, ..., a_5)^T$, and should involve some terms with h in them.

3. Consider the natural cubic spline s on $[0, 2]$ given by

   $$s(x) = \begin{cases} s_0(x) = 1 + 2x - x^3 & \text{if } 0 \le x < 1 \\ s_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{if } 1 \le x \le 2\,. \end{cases}$$

   Find b, c, and d.

4. Repeat Question #3, enforcing the not-a-knot condition instead of the natural cubic spline condition.

5. $\boxed{\textbf{SOURCE CODE PROVIDED:}}$

   - `C = Cubic_Spline(t,y,x)`

Download and modify the provided source code to implement the NATURAL BOUNDARY CONDITIONS.

The provided routine evaluates the cubic spline that fits $(\mathtt{t}, \mathtt{y})$ and evaluates it at all the points provided by `x`. The boundary conditions used by this method are the NOT-A-KNOT boundary conditions. **NOTE:** `x` can be a vector of values, not just a scalar.

6. Let

$$f(x) = \cos(3\pi x) \quad \text{on} \quad x \in [0, 1].$$

Consider the $m + 1$ equally spaced nodes given by

$$x_i = ih \quad i = 0, 1, \ldots, m, \quad h = 1/m.$$

   (a) Use the above code to interpolate $f(x)$ at the nodes x_i with $h = 0.1$. Show a plot of $S(x)$ and $f(x)$ for $x \in [0, 1]$.

   (b) Compare your results from part (a) with the Lagrange interpolant. (You may want to reuse the code you used in the previous homework assignment).

   (c) Interpolate $f(x)$ at the the nodes x_i for h = 0.1, 0.05, 0.025, 0.0125, 0.00625. For each h, calculate the ∞-norm error:

$$\max_{x \in [0,1]} |f(x) - S(x)|.$$

   (d) Based on your results from part (b), what is the convergence rate of this method?

7. Let

$$f(x) = \sin(4\pi x) \quad \text{on} \quad x \in [0, 1].$$

   (a) Use the above code to interpolate $f(x)$ at the nodes x_i with $h = 0.1$. Show a plot of $S(x)$ and $f(x)$ for $x \in [0, 1]$.

   (b) Compare your results from part (a) with the Lagrange interpolant. (You may want to reuse the code you used in the previous homework assignment).

   (c) Interpolate $f(x)$ at the the nodes x_i for h = 0.1, 0.05, 0.025, 0.0125, 0.00625. For each h, calculate the ∞-norm error.

   (d) Based on your results from part (b), what is the convergence rate of this method? Is this result different than for problem #1? Explain.

8. $\boxed{\textbf{ERROR ANALYSIS:}}$

The errors generated by a numerical method on a test problem with various grid resolutions have been recorded in the following table:

GRID SPACING (h)	**ERROR** (E)
5.00000e-02	1.036126e-01
2.50000e-02	3.333834e-02
1.25000e-02	1.375409e-02
6.25000e-03	4.177237e-03
3.12500e-03	1.103962e-03
1.56250e-03	2.824698e-04
7.81250e-04	7.185644e-05
3.90625e-04	1.813937e-05

(a) Make a log-log plot that displays h on the x-axis, and E on the y-axis.

(b) What is the rate of convergence of this system. Demonstrate your conclusions by adding an extra plot to your figure.