Math 451	Homework 2	**Assigned**:	Thursday, September 12
Fall 2013	"Rootfinding"	**Due**:	Friday, September 27

Directions:

- Include a cover page.
- For each problem, submit only the final version of your solution. Each problem should be collocated and stapled.
- Always label every plot (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB (or matplotlib) when comparing 2 or more plots to make comparison easier and to save paper.

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1. Consider the function $f(x) = x^3 + 2\cos(\pi x)$.
   (a) Show that f(x) has a real root α in the interval $[1, 2]$.
   (b) Compute an approximation to α by taking 4 steps of the **bisection method** by hand.
   (c) Repeat, using 2 steps of **Newton's method**, with an initial guess $p_0 = 1.5$.
   For each method, fill in a table of the following form:

n	$p_n^{(Bisect)}$	$f\left(p_n^{(Bisect)}\right)$	$p_n^{(Newton)}$	$f\left(p_n^{(Newton)}\right)$
1				
2				
3			*	*
4			*	*

2. Write separate MATLAB (or SciPy) functions that perform the **bisection method**, the **method of false position**, **Newton's method** and the **secant method**. These functions should be written so that they can be called by typing:

   - [x, NumIters] = Bisection(@f, a, b, TOL, MaxIters)
   - [x, NumIters] = FalsePos(@f, a, b, TOL, MaxIters)
   - [x, NumIters] = Newton(@f, @df, p0, TOL, MaxIters)
   - [x, NumIters] = Secant(@f, p0, p1, TOL, MaxIters)

   Here $f(x)$ is the function whose root we are trying to approximate, and $df(x)$ is the derivative of $f(x)$.

   Use the suggested stopping criteria from the textbook for each method. Your scripts will be graded for correctness as well as completeness of documentation and cleanliness of code.

These codes are to be submitted as part of your assignment, by email. Use the subject line: "MTH 451 HW2 Scripts" in your email.

3. Each of the functions

$$f_1(x) = \sin x - x - 1$$
$$f_2(x) = \frac{1}{2}\ln(x+3) - \sin x$$
$$f_3(x) = e^x - x^2 + 3x - 2$$

has a root in the interval $x \in [-2, 1]$. Use your root-finding scripts to approximate the root of each function, to within an absolute tolerance $TOL = 10^{-6}$. Limit the number of iterations to 500. For Newton's method, use $p_0 = 1$, and for the secant method use $p_0 = 1$ and $p_1 = 0.9$. Summarize the results in a total of three tables, one for each function of the form:

$f_i(x)$	Newton	Secant	False Position	Bisection
Number of Iterations				
Final approximation				

Also comment briefly on the following key points:

(a) Why did the bisection method require approximately the same number of iterations to converge for each test problem?

(b) Newton's method should have experienced difficulty approximating the root of one of the test functions. Which function was this, and what was the difficulty?

(c) Above you used the bisection method to find the root of the function $f_1(x) = \sin(x) - x - 1$. Consider the function $g_1(x) = (\sin(x) - x - 1)^2$. Clearly $f_1(x)$ and $g_1(x)$ have the same root in $x \in [-2, 1]$. Could the bisection method be used to numerically approximate the root of $g_1(x)$? Why or why not?

4. The function $f(x) = \cos(x)$ has a root at $x = \pi/2$. Using the theory we developed for fixed point iterations, find the largest interval around $x = \pi/2$ in which we can choose an initial guess for Newton's method and still be guaranteed to converge to $\pi/2$.

5. A root α is said to be of multiplicity m if $f(x) = (x - \alpha)^m h(x)$, where $h(\alpha) \neq 0$. Show that α is a root of multiplicity m if and only if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0,$$

and $f^{(m)}(\alpha) \neq 0$.

6. Recall that Newtons method is a fixed point iteration where

$$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}$$

Also recall that a sequence $\{p_n\}_{n\geq 0}$ converges to α if

$$\lim_{n\to\infty} \frac{|p_{n+1}-\alpha|}{|p_n-\alpha|^k} = \lambda,$$

where $k > 0$ is the order of convergence and $\lambda > 0$ is the asymptotic error constant. If $k = 1$ then $0 < \lambda < 1$ in order for the sequence to converge linearly.

For each of the following, suppose that α is a root of multiplicity m.

(a) Show that if α is a simple root (i.e. $m = 1$), then Newton's method converges quadratically ($k = 2$) to α.

(b) Show that for all integers $m > 1$ Newtons method converges only linearly ($k = 1$) to α. (NOTE: must show that $0 < \lambda < 1$.)

(c) Consider the following **modified Newtons method:**

$$p_{n+1} = g_m(p_n) = p_n - m\frac{f(p_n)}{f'(p_n)}.$$

Show that this modified Newton's method converges quadratically ($k = 2$) to a root α of multiplicity m.

(d) The function $f(x) = e^{-x^2}\cdot(x-10)^5$ has a root of multiplicity $m = 5$ at $x = 10$. Apply both Newton's method and the modified Newton's method to this problem using an initial guess of $x_0 = 3$ and a tolerance of 10^{-6}. On a single semilogy graph, plot for both methods the absolute error $|\alpha - p_n|$ (y-axis) vs. iteration number n (x-axis).

7. Suppose it was discovered that Commissioner Gordon had the flu when he died, and his core temperature at the time of his death was 103°F. With $k = 0.337114$, solve the equation

$$72 + t_d - \frac{1}{k} + \left(18 + \frac{1}{k}\right)e^{-kt_d} = 103$$

to determine the time of death, t_d based on this new information.

Make clear what method you are using, and what parameters you used.

8. In determining the minimum cushion pressure needed to break a given thickness of ice using an air cushion vehicle, Muller ("Ice Breaking with an Air Cushion Vehicle") in *Mathemaical Modeling: Classroom Notes in Applied Mathematics*, SIAM 1987) derived the equation

$$p^3\left(1-\beta^2\right) + \left(0.4h\beta^2 - \frac{\sigma h^2}{r^2}\right)p^2 + \frac{\sigma^2 h^4}{3r^4}p - \left(\frac{\sigma h^2}{3r^2}\right)^3 = 0,$$

where p denotes the cushion pressure, h the thickness of the ice field, R the size of the air cushion, σ the tensile strength of the ice, and β is related to the width of the ice wedge. Take $\beta = 0.5, r = 40$ feet and $\sigma = 150$ pounds per square inch (psi). Construct a table with h in the first column, and p in the second column for $h = 0.6, 1.2, 1.8, 2.4, 3.0, 3.6$ and 4.2 feet.

Make clear what method you are using, and what parameters you used.