Math 451	Homework 1	**Assigned**:	Tuesday, September 3
Fall 2013	"Taylor series and convergence"	**Due**:	Wednesday, September 11.

Directions:

- Include a cover page.
- Do not hand in scratch work.[1] The final version of your solution to each problem should be collocated and stapled.
- Always label every plot (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB (or matplotlib) when comparing 2 or more plots to make comparison easier and to save paper.

---

1. Compute the first 5 terms in the Taylor series (constant, linear, quadratic, cubic, and quartic pieces) for the following functions about the given point:

   (a) $f(x) = \sin(x)$, about the point $a = \pi/4$.
   (b) $f(x) = x/(1+x)$, about the point $a = 0$. *Hint:* you may want to start with $1/(1+x)$.
   (c) $f(x) = e^{\cos(x)}$, about the point $a = 0$.
   (d) $f(x) = 1 - x - 2x^2 + x^3$, about the point $a = 1$.

2. Using the results from Problem 1(c), make a single MATLAB (or matplotlib) plot which contains all of the following:

   (a) A graph of $f(x) = e^{\cos(x)}$ versus x for $x \in (-3, 3)$.
   (b) A graph of $P_2(x)$.
   (c) A graph of $P_4(x)$.
   (d) A title, x-axis label, y-axis label, and a legend.

3. Determine which one of the following sequences converges to 1 faster (clearly explain your reasoning):
$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} \quad \text{and} \quad \lim_{x\to 0} \frac{\sin^2(x)}{x^2}.$$

4. Compute each of the following limits and determine the corresponding rate of convergence:

   (a) $\lim_{x\to 0} \frac{e^x - 1}{x}$
   (b) $\lim_{x\to 0} \frac{\sin(x)}{x}$
   (c) $\lim_{x\to 0} \frac{\cos(x) - 1 + x^2/2 - x^4/24}{x^6}$

[1]Yes, doing math will require a lot of scratch work! You will rarely be able to correctly answer each question on your first try. For each problem, figure out what the solution is, and then write-up a *final draft* that you hand in.

5. Consider the function $f(x) = e^x$.

   (a) Derive the n^{th} Taylor polynomial $P_n(x)$ as well as the remainder term $R_n(x)$ for the function $f(x)$, expanded about the point $x = 0$.
   (b) Using the remainder term from part (a), determine the value of n needed to guarantee that $|P_n(-1) - f(-1)| < 10^{-5}$.
   (c) Compute the actual error, $\mathrm{err}(-1) = |P_n(-1) - f(-1)|$.

6. Suppose theory indicates that a sequence $\{p_n\}$ converges to p with order 1.5. Explain how you would numerically verify this order of convergence. In particular, demonstrate your method by constructing a table: select an asymptotic error constant $\lambda = 0.5$, and an initial approximation that satisfies $|e_1| = 1$.

7. Consider the recursive sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a}.$$

   (a) Suppose $x = \lim_{n\to\infty} x_n$ exists. Show that $x = 0$ or $x = \pm\sqrt{a}$.
   Hint: If this limit exists, then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$.
   (b) Show that the convergence of this sequence toward $\sqrt{a}$ is third order.
   (c) What is the asymptotic error constant?
   (d) Numerically verify your results by constructing a table similar to the proposed table in the previous problem. For the numerics only, use $a = 1$ and a starting value of $x_0 = 131$. You will only get in a few iterations before hitting machine precision.

8. Consider the functions $f(x) = 1/(1 - x)$ and $g(x) = 1/(1 + x)$.

   (a) Obtain the infinite Taylor series representation of $f(x)$ about the point $x = 0$.
   (b) Obtain the infinite Taylor series representation of $g(x)$ about the point $x = 0$.
   Hint: you may reuse what you computed in Part (a).
   (c) Obtain the infinite Taylor series representation for $\ln(1 + x)$ by identifying this function as

$$\ln(1 + x) = \int_0^x \frac{1}{1 + t}\, dt.$$
