A brief review of Calculus with an eye towards Numerical Analysis

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1 A brief review of Calculus

Recall from a first semester calculus course:

Definition 1 (Derivative)**.** *The* derivative *of a function at a point a is defined by*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{1}$$

Note: an equivalent definition is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{2}$$

In a first semester class, you see that f' is a measure of the rate of change of a function, and provides the *slope* of the tangent line to the curve at the point $(a, f(a))$. Today, we'll formalize this with an approximation method, and in particular, we'll construct devices that tell us precicely how well our approximation performs. In particular, we'll discuss how Taylor series can help improve the accuracy.

1.1 Linear approximation

A first pass at approximating a function comes directly from the definition of the derivative. If we take Definition 1 and take $x \neq a$, but close, we can argue that

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}, \tag{3}$$

which implies

$$f(x) \approx f(a) + f'(a)(x - a). \tag{4}$$

Equation (4) precicely defines an *algorithm* to approximate a function:

1. Input: A point a, the value of the function at that point, $f(a)$ and the function's derivative at that point, $f'(a)$.

2. Ouptut: The coefficients a_1 and a_0 for an equation that *approximates* this function: $y = a_1(x - a) + a_0$.

Remark 1. *A natural question to ask is the following: if we know what f is, then why are we creating something that carries error with it?*

Answer: In numerical analysis, one usually works with finitely many points, and so in practice, you don't have $f(x)$ everywhere. Much of this course will be devoted to answering two questions:

1. Given a discrete set of data, how do we form a continuous approximation to a continuous problem?

2. How do we analyze the error incurred with this approximation, and can we formalize how well our approximations work? Can we *prove* that our method of approximation *converges* to the exact solution?

continuous (e.g. ODE, PDE) $\leftrightarrow$ discrete (e.g. computer arrays).

1.2 Some big theorems

The first big theorem you see in a first semester Calculus course is the following:

Theorem 1 (Mean Value Theorem)**.** *If f is a differentiable function on (a, x), and continuous on $[a, x]$, then there exists (at least) one number $c \in (a, x)$ such that*

$$f'(c) = \frac{f(x) - f(a)}{x - a}. \tag{5}$$

Draw a picture proof of this theorem. Clearly indicate what the endpoints are, and in the picture, show two possible choices for what c can be.

This is precicely the tool that is needed to make our approximation in equation (4) rigorous. The MVT (5) is *exact* but we don't know what c is. However, we can use it anyway as a theoretical tool to show exactly how close (4) approximates the function.

1.3 A rigorous analysis of linear approximation

If we rearrange the result from the MVT, we can identify $f(x)$ by:

$$f_{\text{ex}}(x) = f(a) + f'(c)(x - a), \tag{6}$$

where c is some number between a and x. Comparing this to the linear approximation given by

$$f_{\text{approx}}(x) \approx f(a) + f'(a)(x - a), \tag{7}$$

we see that the only difference is given by the function's derivative. Taking the absolute value of the difference gives us a handle on the error:

$$\begin{aligned} \text{err}(x) = |f_{\text{ex}}(x) - f_{\text{approx}}(x)| &= |f'(c)(x - a) - f'(a)(x - a)| \qquad (8) \\ &= |x - a||c - a| \left| \frac{f'(c) - f'(a)}{c - a} \right|. \qquad (9) \end{aligned}$$

One final observation comes from applying the MVT again[1] , but this time to f', and not f. That is, we know that

$$f''(c^*) = \frac{f'(c) - f'(a)}{c - a}, \tag{10}$$

[1]Here, we need to assume that f' is differentiable and continuous on the closed interval.

for some c^* between a and c. If we insert this into our above estimate, we have,

$$\mathrm{err}(x) = = |x-a||c-a|\,|f''(c^*)| \leq (x-a)^2 M, \tag{11}$$

where M is an upper bound that satisfies $f''(x) \leq M$ for all x near a. Note that as $x \to a$, the error goes to zero, which is exactly what we would like to see!

2 Taylor's theorem

The material presented in the previous section is a special case of a Taylor polynomial, P_1:

$$P_1(x) := f(a) + f'(a)(x-a). \tag{12}$$

Equation (12) defines an exact decomposition of f into:

$$f(x) = P_1(x) + R_1(x), \tag{13}$$

where $|R_1(x)| \leq (x-a)^2 M$ defines the remainder (error).

We know from our error estimates that one way to clamp down on the error is to make x closer to a, but what if we can't do that? On the other hand, what if we have more information about f, such as $f''(a), f'''(a), \ldots$?

Theorem 2 (Taylor's Theorem)**.** *Suppose f is continuous on $[a,b]$, has n continuous derivatives on (a,b) and $f^{(n+1)}$ exists on $[a,b]$. Let $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x and x_0 such that*

$$f(s) = P_n(x) + R_n(x), \tag{14}$$

where the n^{th}-order Taylor polynomial is given by,

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{15}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}. \tag{16}$$

Remark 2. *The only difference between the Remainder, $R_n(x)$, and the next term in the Taylor series is where $f^{(n+1)}$ gets evaluated.*

2.1 An extensive example

Consider the function $f(x) = \sqrt{x}$. We'll construct the second-order Taylor polynomial P_2 of f centered at $x_0 = 16$. This will decompose f into $f(x) = P_2(x) + R_2(x)$.

n	$f^{(n)}$	$f^{(n)}(x_0 = 16)$	$f^{(n)}(x_0 = 16)/n!$
0	$\sqrt{x}$	4	4
1	$1/(2\sqrt{x})$	$1/8$	$1/8$
2	$-1/(4x^{3/2})$	$-1/256$	$-1/512$
3	$3/(8x^{5/2})$	$\cdots$	$\cdots$

A *quadratic* approximation to the function is then given by

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2! \tag{17}$$

$$= 4 + \frac{1}{8}(x - 16) - \frac{1}{512}(x - 16)^2, \tag{18}$$

with a *remainder* given by

$$R_2(x) = \frac{1}{16\xi^{5/2}}(x - 16)^3. \tag{19}$$

We can use this polynomial to approximate stuff nearby. For example,

$$\sqrt{17} = f(17) =\approx P_2(17) = 4 + \frac{1}{8} - \frac{1}{512} = 4.123046875.$$

The error incurred by using this for our approximation is bounded by

$$|R_2(17)| = \left|\frac{f^{(3)}(\xi)}{3!}(x - x_0)^3\right| = \left|\frac{1}{16\xi^{5/2}}\right| < \frac{1}{1616^{5/2}} = 1/16384 \approx 6.10 \times 10^{-5}. \tag{20}$$

2.2 Taylor polynomial of a polynomial

Consider the function $f(x) = 5 - 2x + 3x^2 + x^3$. Here, we'll construct the Taylor polynomial for this function centered at $x_0 = 4$.

2.3 Other examples

If you have time, do a table for a couple of other Taylor series:

$$e^x, \quad \sin(x), \quad \cos(x), \quad 1/(1 - x). \tag{21}$$

3 Multivariable Taylor series

If you have time, you may want to write down the first and second Taylor polynomial in the multivariable case.