NOTES ON SEQUENCES AND SERIES

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A *sequence* is a function, $a : \mathbb{N} \to \mathbb{R}$, whose domain is a discrete set of points. We normally denote the sequence with subscripts, using $a(n) = a_n$ in place of $a(n)$. Other means of writing a sequence including using "set" notation:

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}, \tag{1}$$

which should *not* be confused with sets, because while order matters for a sequence, order certainly does not matter for a set. We rarely concern ourselves with the starting index. That is, $\{a_6, a_7, \dots\}$ is also a sequence. That sequence has a starting index of 6 instead of 1.

The big theorem which will get a lot of mileage is the following:

Theorem 1 (Finite Mountain Climber)**.** *If* $\{a_n\}$ *is a* bounded*,* monotonic *sequence, then* $\lim_{n\to\infty} a_n$ *exists.*

Picture a mountain climber with perfect grip. She never falls, nor gets tired, which means the sequence is *non-decreasing*: $a_0 \le a_1 \le a_2 \le \dots$. If she's climbing a mountain with finite height (i.e. $a_n \le M$, for all n), then she must eventually settle down at some point on the mountain. It might not be the peak, perhaps she found an awesome lake 1K feet from the top to hang out at, but she will settled down at some altitude.

A *series* is formed by taking a sequence, and adding up every term in the sequence. Formally, this is the single number, given by

$$\sum_{n=0}^{\infty} a_n. \tag{2}$$

Again, we rarely care about the lower index of summation, because we're more interested in whether or not the infinite sum *converges*. If we change the lower index on a convergent series, then the new series is identical to the old one, but off by a constant. This means $\sum_{n=6}^{\infty} a_n$ is also considered a series.

An infinite sum is potentially ambiguous and therefore we need to formalize what we mean by adding up infinitely many numbers. In order to determine if the sequence *converges*, (i.e. gives us an actual number), we form a second sequence, called the *sequence of partial sums*, which is defined as:

$$S_n = \sum_{k=0}^{n} a_k. \tag{3}$$

Each S_n is well defined, because its a sum of finitely many terms, and therefore, the sequence $\{S_n\}_{n=0}^{\infty}$ is well defined. We say that the series defined in equation (2) *converges* if and only is the sequence $\lim_{n\to\infty} S_n$ converges. Remember, there are exactly two sequences to consider when looking at a series.

(1) Given a sequence $\{a_n\}_{n=0}^{\infty}$, we can form:
(2) The sequence of partial sums: $\{S_n\}_{n=0}^{\infty}$, where each term in this sequence is defined by

$$S_n := a_0 + a_1 + \cdots a_n = \sum_{k=0}^{n} a_k.$$

(3) If the sequence of partial sums, $\{S_n\}_{n=0}^{\infty}$ converges, then we say that the *infinite series* $\sum a_n$ *converges.*

The primary question we will attempt to answer throughout the bulk of Chapter 10 is the following: given a sequence $\{a_n\}$, under what conditions does the series $\sum a_n$ converge?

The first tool to throw in your toolbox is the following:

Theorem 2 (n^{th} Term (or No Way!) Test)**.** *If* $\lim a_n \neq 0$*, then the series* $\sum a_n$ *diverges.*

As an application of the n^{th}-term test, consider the following series: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$. Because $\lim \ln\left(\frac{n}{2n+1}\right) = \ln\left(\lim \frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0$, the series diverges by the n^{th}-term test. Roughly, the statement of this theorem means is that towards the tail end of the series, we are consistently adding in a "large", non-zero amount, infinitely many times.

A very special series that we'll see plenty of is the geometric series, which converges if and only if the common ratio r is less than 1 in absolute value:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}. \tag{4}$$

Your textbook may have a slightly different version of this written down, but you can always force *every* geometric series to look like this by first factoring out the first term in the series.