NOTES ON REVIEWING FOR FINAL EXAM

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6. Applications of Definite Integrals

There are a number of ways to generate a 3D object from a 2D domain. Usually, you're given some formula, or access to some formula $A(x)$ which represents the area of a cross-sectional slice. In general, we define the volume of an object to be

$$V = \int_a^b A(x)\, dx. \tag{1}$$

An example of a generic problem would be the following:

Example 1. *Suppose the base of an object is the region $x^2 + y^2 \leq 1$. Vertical slices of the object produce isocoles right triangles with one leg on the base of the object. Compute the volume of the object.*

A special class of problems that we looked at are when the cross-sectional areas look like discs.

In this case, the *Disc method* has a cross-sectional area given by $A(s) = \pi r^2$, which gives,

$$V = \int \pi r^2\, ds. \tag{2}$$

For vertical slices, $s = x$, and for horizontal slices $s = y$. Of course, limits of integration also need to be in place.

Two applications of the disc method produces the *washer method*:

$$V = \pi \int r_{out}^2 - r_{in}^2\, ds. \tag{3}$$

Example 2. *Compute the volume of the object generated by revolving the region bounded by $x = \sqrt{5}y^2, x = 0, y = -1, y = 1$ about the y-axis. (See 6.1.27).*

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Arclength should look familiar from the recent material. When $y = y(x)$, the formula looks like,

$$L = \int_c^d \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx, \tag{4}$$

and when $x = x(y)$, the formula swaps y with x:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy, \tag{5}$$

See 6.3, problems $\{1, 2, \ldots, 10\}$.

For constant force and constant distance, we define *work* as

$$W = F \cdot d. \tag{6}$$

In general, we define it as

$$W = \int_a^b F(x)\, dx. \tag{7}$$

A few problemg to be aware of include variations of Hooke's law, work required to pull a rope up the side of a building, or work required to pump out the contents of a container.

See 6.5 problems $\{2, 7, 9, 13, 17\}$ and other problems from that section.

7. Chapter 7

Know the horizontal line test, the definition of a $1-1$ function and how to find an inverse to a function graphically or algebraically if it exists. The biggest result from 7.1 is the *Derivative Rule for Inverse Functions*:

Theorem 1. *If f has an open interval I for its domain, and $f'(x)$ exists on I and is never zero, then*

(1) *the inverse function, f^{-1} exists, and*
(2) *the derivative of f^{-1} is given by*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}. \tag{8}$$

For practice, see problems $\{25, \ldots, 34\}$ from 7.1.

Know how we define $\ln(x)$, and more importantly, what its derivative is. Know the algebraic properties of $\ln(x)$: product rule, quotient rule, recipricol rule and power rule. The product rule is

$$\ln(AB) = \ln(A) + \ln(B).$$

Logarithmic differentiation can be very useful in some cases to avoid excessive use of quotient/chain rule.

Example 3. *Compute* $\frac{dy}{dx}$, *where*

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{3/2}}.$$

We then defined e^x as the inverse function of $\ln(x)$. It has a derivative, and its own set of multiplicative, division and recipricol rules.

Remember that we defined

$$a^x = e^{a\ln(x)}, \tag{9}$$

and therefore $\frac{d}{dx}a^x = a^x\ln(a)$. After defining a, we get the change of base formula for ln:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}. \tag{10}$$

We looked at how to solve a class of differential equations, that are called *separable*. Look at any of $\{9, 10, \ldots, 22\}$ from 7.4. The most important models we looked at were

(1) Population Growth:

$$\frac{dy}{dx} = ky, \quad k > 0.$$

(2) Radioactive Decay:

$$\frac{dy}{dx} = ky, \quad k > 0$$

(3) Newton's Law of Cooling:

$$\frac{dy}{dt} = -k\left(y - y_s\right).$$

where y_s is the surrounding temperature, and $k > 0$ is a constant.

Know how to solve all of these. See 7.4 problems $\{25, 29, 38, 39\}$.

In section 7.5, we learned LeHópital's rule. Under appropriate conditions, it says that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \tag{11}$$

provided both $f, g \to 0$ as $x \to a$. Practice many as many problems from 7.5 as necessary $\{7, \ldots, 50\}$, any. The second class you'll run into is variations of the rule such as the following:

Example 4. *Compute the following limit:*

$$\lim_{x \to 0^+} x^x.$$

Here, you need to first take logarithms first. For more practice on this class of problem, see 7.5 $\{51, \ldots, 66\}$, any.

Know the following three inverse trig functions:

(1) $\sin^{-1}(x)$,
(2) $\tan^{-1}(x)$,
(3) $\tan^{-1}(x)$.

Know what their graphs look like, but most importantly, what their derivatives are. In order to compute the derivative of $y = \sin^{-1}(x)$, you can first apply sin to both sides:

$$\sin(y) = x,$$

then differentiate to get:

$$\cos(y)y' = 1, \implies y' = \sec(x) = \frac{1}{\sqrt{1-x^2}}.$$

Look up some of the *hyperbolic* functions. I think the easiest way to always deal with these is to simply convert them to exponentials, so memorize their integrals if you must, otherwise know how to convert them!

8. Chapter 8

Look up table 8.1. You most likely will not need to have all of these memorized, but at a minimum, I would memorize $1-9$, $12-15$, and $18-20$.

Theorem 2. *Integration by parts is best memorize by*

$$\int u\, dv = uv - \int v\, du. \tag{12}$$

There are three very common problems that IBP can be used to solve:

(1) polynomial times sin, cosine, or exponential.
(2) sin, cosine, exponential times sin, cosine or exponential.
(3) integral of an inverse function where you know the derivative.

For an example of (1), consider

Example 5. *Compute*

$$\int (2x^2 + x - 5)\sin(2x)\, dx.$$

Tabular integration knocks the socks off these problems.

For an example of (2), consider

Example 6. *Compute*

$$\int e^{2x} \sin(x)\, dx.$$

For an example of (3), consider

Example 7. *Compute*

$$\int \ln(x), dx.$$

or rather

Example 8. *Compute*

$$\int \tan^{-1}(x), dx.$$

Certain tri functions can be integrated with an appropriate use of trig substitutions. Two of your best friends are

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right), \tag{13}$$

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right), \tag{14}$$

and pythagorian theorem,

$$\sin^2(x) + \cos^2(x) = 1. \tag{15}$$

Know how to deal with stuff of the form: $\int \cos^m(x) \sin^n(x)\, dx$ and variations of that.

Trig subs can also be useful. The most common subs are given by

1. If you see something with $a^2 - x^2$ use $x = a\sin(\theta)$.
2. If you see something with $a^2 + x^2$ use $x = a\tan(\theta)$.
3. If you see something with $x^2 - a^2$ use $x = a\sec(\theta)$.

The whole point is stuff like $\sqrt{a^2 - x^2}$ will collapse into a perfect square with said substitution.

See any problems $\{15, \dots 48\}$ from 8.4 for practice.

Another class of problems that we can integrate, which don't appear in table 8.1, are *rational functions*. In principle, we can integrate every rational function after doing a PFD (partial fraction decomposition) on it. **Warning:** do *not* attempt to do a PFD on a function that's not a rational function, such as

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

The following are all rational functions:

$$\frac{1}{x^2+2x+1}, \quad \frac{x^2+1}{x(x+2)}, \quad \frac{3x+1}{x^3+5x+2}, \quad \dots$$

See problems from 8.4.

Know when you're dealing with an *improper integral*, and how to deal with them. Two common things can happen: either the function has an asymptote somewhere within the limits of integration, or one of the limits (or both) are infinite. The following are two examples,

Example 9. *Compute the following integral,*

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx$$

Example 10. *Compute the following integral,*

$$\int_0^5 \frac{x}{x^2-2x+1}\,dx$$

We also have the continuous version of the Limit Comparison Test, as well as the Direct Comparison Test. See theorems 2 and 3 from §8.7.

10. Chapter 10

A *sequence* is a function, $a : \mathbb{N} \to \mathbb{R}$, whose domain is a discrete set of points. We normally denote the sequence with subscripts, using $a(n) = a_n$ in place of $a(n)$. Other means of writing a sequence including using "set" notation:

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}, \tag{16}$$

which should *not* be confused with sets, because while order matters for a sequence, order certainly does not matter for a set. We rarely concern ourselves with the starting index. That is, $\{a_6, a_7, \dots\}$ is also a sequence. That sequence has a starting index of 6 instead of 1.

A *series* is formed by taking a sequence, and adding up every term in the sequence. Formally, this is the single number, given by

$$\sum_{n=0}^{\infty} a_n. \tag{17}$$

Again, we rarely care about the lower index of summation, because we're more interested in whether or not the infinite sum *converges*. If we change the lower index on a convergent series, then the new series is identical to the old one, but off by a constant. This means $\sum_{n=6}^{\infty} a_n$ is also considered a series.

An infinite sum is potentially ambiguous and therefore we need to formalize what we mean by adding up infinitely many numbers. In order to determine if the sequence *converges*, (i.e. gives us an actual number), we form a second sequence, called the *sequence of partial sums*, which is defined as:

$$S_n = \sum_{k=0}^{n} a_k. \tag{18}$$

Each S_n is well defined, because its a sum of finitely many terms, and therefore, the sequence $\{S_n\}_{n=0}^{\infty}$ is well defined. We say that the series defined in equation (17) *converges* if and only is the sequence $\lim_{n\to\infty} S_n$ converges. Remember, there are exactly two sequences to consider when looking at a series.

(1) Given a sequence $\{a_n\}_{n=0}^{\infty}$, we can form:
(2) The sequence of partial sums: $\{S_n\}_{n=0}^{\infty}$, where each term in this sequence is defined by

$$S_n := a_0 + a_1 + \cdots a_n = \sum_{k=0}^{n} a_k.$$

(3) If the sequence of partial sums, $\{S_n\}_{n=0}^{\infty}$ converges, then we say that the *infinite series* $\sum a_n$ *converges.*

The primary question we will attempt to answer throughout the bulk of Chapter 10 is the following: given a sequence $\{a_n\}$, under what conditions does the series $\sum a_n$ converge?

A very special series that we'll see plenty of is the geometric series, which converges if and only if the common ratio r is less than 1 in absolute value:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}, \quad |r| < 1. \tag{19}$$

Every other series requires a test to be performed to see if it converges. Here is a list of most of the tests we encountered:

Theorem 3 (n^{th} Term (or No Way!) Test)**.** *If* $\lim a_n \neq 0$, *then the series* $\sum a_n$ *diverges.*

Example 11. *Determine whether or not the following series converges,*

$$\sum_{n=0}^{\infty} \cos(\pi n)$$

Example 12. *Determine whether or not the following series converges,*

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$$

Theorem 4 (Integral Test)**.**

Example 13. *Determine whether or not the following series converges,*

$$\sum_{n=1}^{\infty} \frac{1}{n\left(\ln(n)\right)^2}$$

Theorem 5 (Direct Comparison Test)**.**

Example 14. *Determine whether or not the following series converges,*

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1}$$

Theorem 6 (Limit Comparison Test)**.**

Example 15. *Determine whether or not the following series converges,*

$$\sum_{n=0}^{\infty} \frac{n^{1/3}}{\sqrt{n^2+1}}$$

Theorem 7 (Ratio Test)**.**

You should think about applying the Ratio Test on essentially every problem you see that has a factorial in it.

For problems with negative terms, first check if it fails the n^{th}-term test, if it passes it (i.e. the test is inconclusive because $\lim a_n = 0$, then check to see if it *converges absolutely*. If it doesn't converge absolutely, check to see if you can apply the

Theorem 8 (Alternating Series Test (Leibniz Test))**.** *Suppose you have a sequence of the form* $\sum(-1)^n a_n$*, where each* $a_n \geq 0$*. If the following occur,*

(1) *The sequence* $\{a_n\}$ *is* non-increasing*, i.e.* $a_{n+1} \leq a_n$*, and*
(2) $\lim a_n = 0$*,*

then the series converges (at least conditionally).

To determine absolute convergence, you need to look at

$$\sum |(-1)^n a_n| = \sum a_n.$$

Know how to add, subtract, multiply differentiate and integrate power series. Know the definitions of Taylor series, Taylor polynomials and how to use them.

Theorem 9 (Error Estimation Theorem)**.** *Suppose $f(x)$ is sufficiently smooth, and that*

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots a_n(x-a)^n$$

is the n^{th}-degree Taylor polynomial, where $a_k = \frac{f^{(k)}(a)}{k!}$. Then the error, $R_n(x) = f(x) - P_n(x)$ satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} c^{n+1},$$

where c is some number between x and a. Furthermore, if $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

11. Chapter 11

Material from this chapter should be somewhat fresh.

Paramterizations of curves, plotting and graphing curves in Polar coordinates, changing from Cartesian to Polar and back again. Area in polar coordinates:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r(\theta)^2 \, d\theta. \tag{20}$$