Exam 4

Math 133
December 4^{th}, 2012

Name: Key

Question	Points	Your Score
Q1	20	
Q2	15	
Q3	15	
Q4	15	
Q5	20	
Q6	15	
TOTAL	100	

Read all of the following information before starting the exam:

- Show all work, clearly and in order. "Answers" without justification will receive zero credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: `https://www.msu.edu/~ombud/academic-integrity/index.html`. Simply put, don't cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!

Problem 1 (20 points) Determine the largest open interval I for which the following Power series converge. If the series converges at a single point only, indicate which point.

(a)

$$\sum_{n=0}^{\infty} \frac{2^{n+1}(x-5)^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+2}}{(n+1)!} \cdot \frac{n!}{2^{n+1}} \right| \cdot \frac{|x-5|^{n+1}}{|x-5|^n}$$

$$= |x-5| \cdot \lim_{n\to\infty} 2 \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 2|x-5| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

$I = (-\infty, \infty)$, converges everywhere.

(b)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(x-2)^n$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{3^{n+1}} \cdot (x-2)^{n+1} \cdot \frac{3^n}{(-1)^n} \frac{1}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-2|}{3} = \frac{|x-2|}{3} < 1$$

Converges where $|x-2| < 3$

i.e. $-3 < x-2 < 3$

$-1 < x < 5$

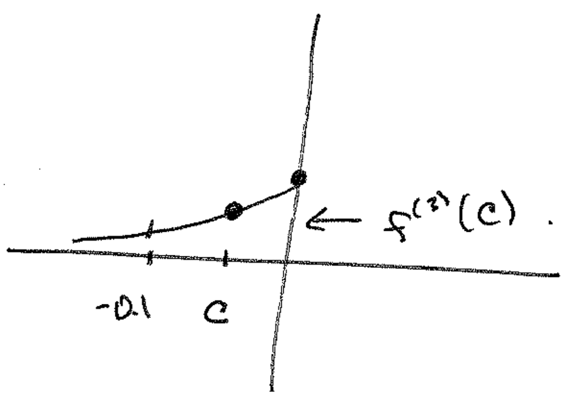
$I = (-1, 5)$.

Problem 2 (15 points) The polynomial, $p(x) = 1 + x + \frac{x^2}{2!}$ is a good approximation to $f(x) = e^x$ for x near zero. Compute an upper bound for the error, $R = |p(-0.1) - f(-0.1)|$ incurred when using p to approximate f at $x = -0.1$.

$$|R| = \left| \frac{f^{(3)}(c)}{3!} (-0.1)^3 \right|$$

$$= \frac{10^{-3}}{3!} |f^{(3)}(c)|, \quad c \in (-0.1, 0).$$

$$f^{(3)}(x) = e^x, \quad \text{so} \quad |f^{(3)}(c)| < \underline{1}.$$



$$\therefore \quad |R| = \frac{10^{-3}}{3!} |f^{(3)}(c)| < \frac{10^{-3}}{3!}$$

Problem 3 (15 points)

(a) Find the Taylor series centered at $x = 0$ (i.e. the Maclaurin series) of the function

$$f(x) = \frac{2x}{(1-x)^2}.$$

Hint: It might be useful to start with a series you already know ...

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\Rightarrow \frac{2x}{(1-x)^2} = 2x + 4x^2 + 3\cdot 2x^2 + 4\cdot 2x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} 2n x^n = \sum_{n=1}^{\infty} 2n x^n .$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

(b) Suppose $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n$. Find $f^{(2012)}(0)$, the 2012^{th} derivative of f evaluate at $x = 0$.

$$a_{2012} = \frac{(-1)^{2012}}{2^{2012}(2012)!} = \cancel{(2012)!}\ \frac{f^{(2012)}(0)}{(2012)!}.$$

$$\Rightarrow f^{(2012)}(0) = \frac{1}{2^{2012}}$$

Problem 4 (15 points) Suppose a curve is parameterized with the following definition,

$$\begin{cases} x(t) = 2\cos(3t), \\ y(t) = 6t + 2\sin(3t), \quad 0 \le t \le \pi/3. \end{cases}$$

Find the length of the curve.

$X' = -6\sin(3t)$. $\quad y' = 6 + 6\cos(3t)$.

$(x')^2 + (y')^2 = 36\sin^2 3t + 36 + 2\cdot 36\cos 3t + 36\cos^2 3t$

$= 36 + 36 + 2\cdot 36\cos(3t)$

$= 2\cdot 36\left(1 + \cos(3t)\right)$

Use :

~~[illegible]~~ $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

so $2\cos^2(\theta) = 1 + \cos(2\theta)$. $2\theta = 3t$:

$(x')^2 + (y')^2 = 2\cdot 36\left(2\cos^2\left(\frac{3t}{2}\right)\right)$

$L = \int_0^{\pi/3} \sqrt{2\cdot 36\cdot 2\cos^2\left(\frac{3t}{2}\right)}\, dt = \int_0^{\pi/3} 2\cdot 6 \left|\cos\left(\frac{3t}{2}\right)\right| dt$

$= 12\int_0^{\pi/3} \cos\left(\frac{3t}{2}\right) dt = 12\cdot\frac{2}{3}\sin\left(\frac{3t}{2}\right)\Big|_0^{\pi/3}$

$= \cancel{12}^{4}\cdot\frac{2}{\cancel{3}}\left(\cancel{\sin\left(\frac{3\cdot\pi}{2\cdot\cancel{3}}\right)}^{1} - \cancel{\sin(0)}^{0}\right)$

$= 8$.

Problem 5 (20 points) Determine whether or not the following series converge.

(a)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$$

$$\lim_{n\to\infty} a_{n+1} \cdot \frac{1}{a_n} = \lim_{n\to\infty} \frac{(2(n+1))!}{\cancel{(n+1)!}\,(n+2)!} \cdot \frac{n!\,\cancel{(n+1)!}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!} \cdot \cancel{n!}}{(n+2)(n+1)\cancel{n!}\,\cancel{(2n)!}}$$

$$= 4.$$ Diverges by Ratio test.

(b)

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = 2 \cdot \lim_{n\to\infty} \left(\frac{n}{(n+1)}\right)^3$$

$$= 2 \cdot \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^3 = 2.$$

Diverges by Ratio test.

(c) True or False. If $\sum a_n$ converges, then $\sum |a_n|$ converges. If True, indicate why. If False, provide a counter-example.

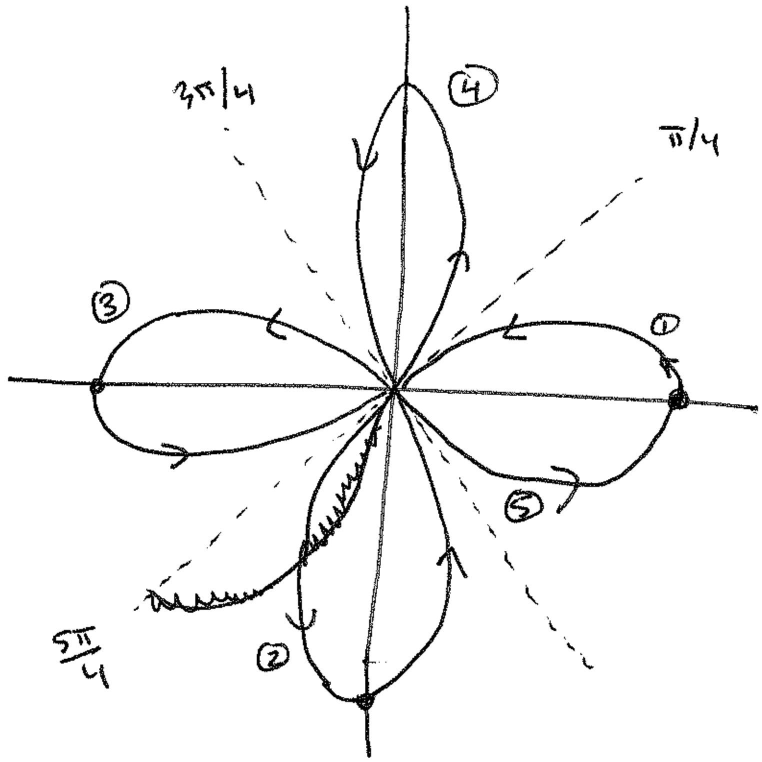
False. $\sum \frac{(-1)^n}{n}$ converges, but $\sum \left|\frac{(-1)^n}{n}\right|$ diverges.

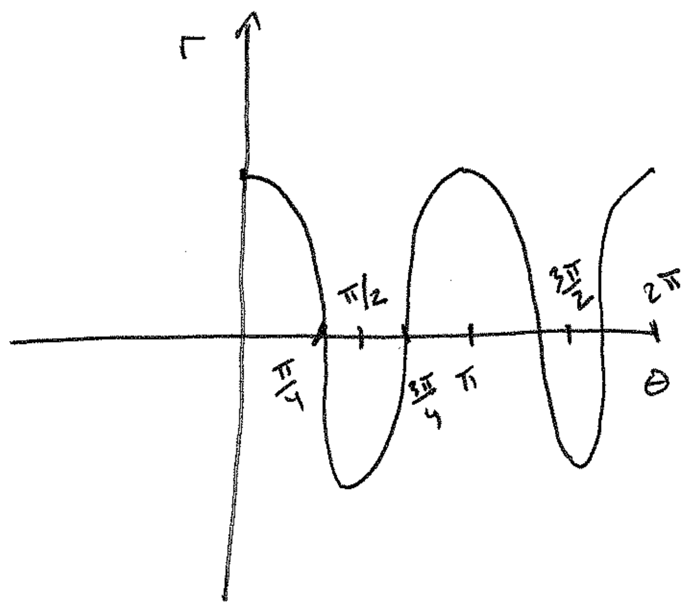
Problem 6 (15 points) Sketch a graph of the curve defined by

$$r(\theta) = 6\cos(2\theta), \quad 0 \le \theta \le 2\pi.$$

Indicate which direction the curve traverses as θ increases.

θ	r	
0	6	1
$\frac{\pi}{4}$	$6 \cdot \cos(\pi/2) = 0$	
$\frac{\pi}{2}$	$6 \cdot \cos(\pi) = -6$	2
$\frac{3\pi}{4}$	0	
π	6	3
$\frac{5\pi}{4}$	0	
$\frac{3\pi}{2}$	$6 \cdot \cos(3\pi) = -6$	4
$\frac{7\pi}{4}$	0	
2π	6	5





Bonus: (10 points - no partial credit)
Consider the function,

$$f(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k,$$

where the *binomial coefficients* are defined as

$$\binom{m}{0} := 1, \quad \binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!}, \quad k \geq 1.$$

(a) Show that

$$f'(x) = \frac{mf(x)}{1+x}, \quad -1 < x < 1.$$

Hint: It "may" be useful to first show,

$$(k+1)\binom{m}{k+1} + k\binom{m}{k} = m\binom{m}{k}.$$

(b) Define $g(x) = (1+x)^{-m} f(x)$. Show that $g'(x) = 0$.

(c) Using part (b), show that $f(x) = (1+x)^m$.

(a)

$$f'(x) = \sum_{k=0}^{\infty} k \binom{m}{k} x^{k-1} = \sum_{k=0}^{\infty} (k+1) \binom{m}{k+1} x^k.$$

$$(1+x) f'(x) = \sum_{k=0}^{\infty} (k+1) \binom{m}{k+1} x^k + x \sum_{k=0}^{\infty} k \binom{m}{k} x^{k-1}$$

$$= \sum_{k=0}^{\infty} \left[(k+1)\binom{m}{k+1} + k \binom{m}{k} \right] x^k.$$

$\therefore$ Provided

$$(k+1)\binom{m}{k+1} + k\binom{m}{k} = m \cdot \binom{m}{k}, \qquad (*)$$

we have $(1+x) f'(x) = m f(x)$, which is part (a).

(b) $g(x) = (1+x)^{-m} f(x)$.

$$g'(x) = -m(1+x)^{-m-1} f(x) + (1+x)^{-m} f'(x)$$

$$= \frac{-m}{(1+x)} g(x) + (1+x)^{-m} \cdot \frac{m f(x)}{(1+x)} \qquad (\text{by (a)})$$

$$= \frac{-m}{(1+x)} g(x) + \frac{m g(x)}{1+x} = 0.$$

(this page intentionally left blank)

(c) From (b), we know $g'(x) = 0$.

$\Rightarrow \quad g^*(x) = \text{constant}.$

but $g(0) = f(0) = 1$, so

$1 = g(x) = (1+x)^{-m} f(x)$

$\Rightarrow \quad \boxed{f(x) = (1+x)^m}$

Pf of (*)

$$(k+1)\binom{m}{k+1} = (k+1)\left(\frac{m(m-1)\cdots(m-k+1)(m-(k+1)+1)}{(k+1)!}\right)$$

$$= \cancel{(k+1)} \cdot \frac{(m)(m-1)\cdots(m-k+1)}{k!} \cdot \frac{(m-k)}{\cancel{k+1}}$$

$$= \binom{m}{k} \cdot (m-k).$$

$$\therefore \quad (k+1)\binom{m}{k+1} + k\binom{m}{k} = \binom{m}{k} \cdot \{(m-k) + k\}$$

$$= m \cdot \binom{m}{k}.$$

(this page intentionally left blank)