Exam 3

Math 133
November 6^{th}, 2012

Name: Key

Older version

Question	Points	Your Score
Q1	~~20~~	
Q2	~~15~~	
Q3	~~15~~	
Q4	~~20~~	
Q5	~~15~~	
Q6	~~15~~	
TOTAL	100	

Problems have been changed by hand to match new version.

Read all of the following information before starting the exam:

- Show all work, clearly and in order. "Answers" without justification will receive zero credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: `https://www.msu.edu/~ombud/academic-integrity/index.html`. Simply put, don't cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!

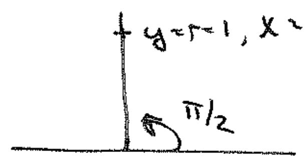
Problem 1 (20 points) For each of the following integrals, determine if they converge or diverge:

(a)

$$\int_0^{1} \frac{1}{\sqrt{1-x^2}}\,dx. = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} \sin^{-1}(x)\Big|_0^b$$

$$= \lim_{b\to 1^-} \left[\sin^{-1}(b) - \sin^{-1}(0) \right] = \sin^{-1}(1)$$

$$= \cancel{\tfrac{\pi}{4}}\ \frac{\pi}{2}$$



(b)

$$\int_0^\infty \frac{1}{2x+\cos(5x)+x^3}\,dx.$$

$$2x + \cos(5x) + x^3 \geq 2x - 1 + x^3 \geq x^3 - 1.$$

$$\Rightarrow \quad \frac{1}{x^3-1} \geq \frac{1}{2x+\cos(5x)+x^3}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^3}}{\frac{1}{x^3-1}} = \lim_{x\to\infty} \frac{x^3-1}{x^3} = \underline{1}$$

So $\displaystyle\int_{10}^{\infty} \frac{1}{x^3-1}\,dx$ converges. by LCT.

By DCT, $\displaystyle\int_{10}^{\infty} \frac{1}{2x+\cos 5x + x^3}\,dx$ converges,

and therefore $\displaystyle\int_0^{\infty} \frac{1}{2x+\cos 5x + x^3}\,dx$ converges.

(b/c $2x+\cos 5x + x^3 > 0$, so has no asymptotes)

Note: Can also use LCT w/ $1/x^3$.

Problem 2 (15 points) Determine whether or not each of the following series converges or diverges:

Note: This series should have started w/ $n=2$.

(a)

$$\cancel{\sum_{n=1}^{\infty} \frac{3}{n(\ln(n))^{1.01}}} \qquad \sum_{n=2}^{\infty} \frac{3}{n(\ln n)^6}$$

Use integral test with $f(x) = \dfrac{1}{x(\ln x)^6}$. Note,

$$f'(x) = \cancel{x(\ln x)^6} - \frac{\left((\ln x)^6 + x \cdot 6(\ln x)^5 \cdot \frac{1}{x}\right)}{x^2(\ln x)^{12}} < 0$$

so f is decreasing and continuous.

$$\int_2^{\infty} \frac{3}{x(\ln x)^6}\,dx = \int_{\ln 2}^{\infty} \frac{3}{u^6}\,du \quad \text{converges,}$$

$u = \ln x$
$du = \frac{1}{x}dx$

So by integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n(\ln n)^6}$ converges.

(b)

$$\sum_{n=1}^{\infty} n \tan\left(\frac{1}{n}\right).$$

$$\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n}$$

$$\overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{\sec^2(1/n)\,\cancel{(-1/n^2)}}{\cancel{(-1/n^2)}} = 1$$

by n^{th}-term test, $\sum n\tan(1/n)$ diverges.

Problem 3 (15 points) Consider the sequence $\{a_n\}$ whose n^{th}-term is given by

$$a_n = \left(1 + \frac{3}{n}\right)^{\frac{1}{2n}}$$

(a) Does the sequence $\{a_n\}$ converge? If yes, find its limit.

Take logs:

$$\ln(a_n) = \frac{1}{2n} \ln(1 + 3/n).$$

$$\lim_{n\to\infty} \ln(a_n) = 0 \qquad \left(\frac{0}{\infty} = 0\right)$$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln(a_n)} = e^0 = 1.$$

By n^{th}-term test, series diverges.

(b) Does the series $\sum a_n$ converge?

No

Problem 4 (20 points) ~~Determine a value for the following infinite sum:~~

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{5^n}.$$

Wording was changed for final version of exam.

~~Does the series converge?~~ ~~Word...~~

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{5^n} = 3 \sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n$$

$$= 3 \cdot \frac{1}{1-(-3/5)} = 3 \frac{1}{1+3/5}$$

$$= 3 \cdot \frac{5}{5+3} = \boxed{\frac{15}{8}}$$

Series converges to $\frac{15}{8}$. (Geo series, $r = \frac{-3}{5}$ and $|r| < 1$)

Problem 5 (15 points)

The partial sums $\{S_n\}_{n=1}^{\infty}$ of a series $\sum_{k=1}^{\infty} a_k$ are given by $S_n = 1 - 1/n$.

(a) Does the series converge?

(Make sure to indicate why or why not - credit will not be given for a yes/no answer.)

Yes. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \underline{1}$.

By definition,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \underline{1}$$

(b) Find the fourth term, a_4 in the sequence $\{a_k\}_{k=1}^{\infty}$.

$$a_4 = S_4 - S_3 = (1 - 1/4) - (1 - 1/3)$$

$$= -\frac{1}{4} + \frac{1}{3} = \frac{-3+4}{12} = \frac{1}{12}.$$

Problem 6 (15 points) Rank each of the following sequences from large to small. Specifically, we say that $\{a_n\}$ is *smaller* than $\{b_n\}$ if and only if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

(a) $a_n = \frac{n^3+n+3}{5n+2}$.

(b) $a_n = n!$

(c) $a_n = \ln(n+1)$

(d) $a_n = n^{0.01}$

(e) $a_n = 2^n + 21n^{2012}$

(f) $a_n = 9e^{2n}$.

(g) $a_n = n\ln(n)$.

(h) $a_n = n^n$

These were changed for final version.

(a) $a_n = \frac{10n^2+3}{n+2} \approx \frac{10n^2}{n} = 10n$

(b) $a_n = n!$

(c) $a_n = 2n^2 + n\ln(n)$

(d) $a_n = n^n$

$\underbrace{n^n >> n!}_{\text{See class notes for why this is true.}} >> \underbrace{2n^2+n\ln(n)}_{\text{fastest part: } 2n^2} >> \underbrace{\frac{10n^2+3}{n+2}}_{\text{grows like } 10n.}$.

Insert the letters (a) – (h) into the following table. 1 = fastest growing sequence, 2 is the second fastest and so on.

1	2	3	4	5	6	7	8

Bonus: True or False: if yes, show this is true, if false, present a valid counterexample.

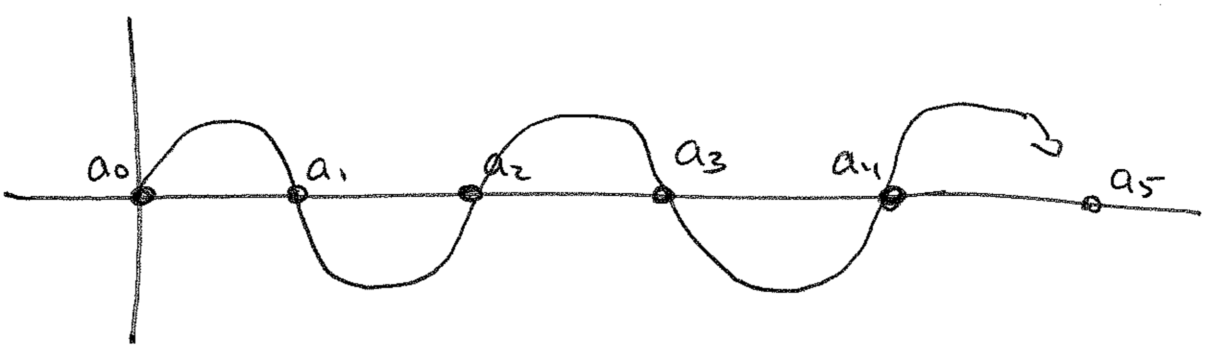
For each part, suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function, and $\{a_n\}$ is a sequence which satisfies $f(n) = a_n$ for each natural number n.

(a) (5 points - all or nothing) If $\sum_{n=0}^{\infty} a_n$ converges, does $\int_0^{\infty} f(x)\,dx$ converge?

No. (see class notes). Take $f(x) = \sin(2\pi x)$.

Then $a_n = f(n) = 0$ so $\sum 0$ converges.

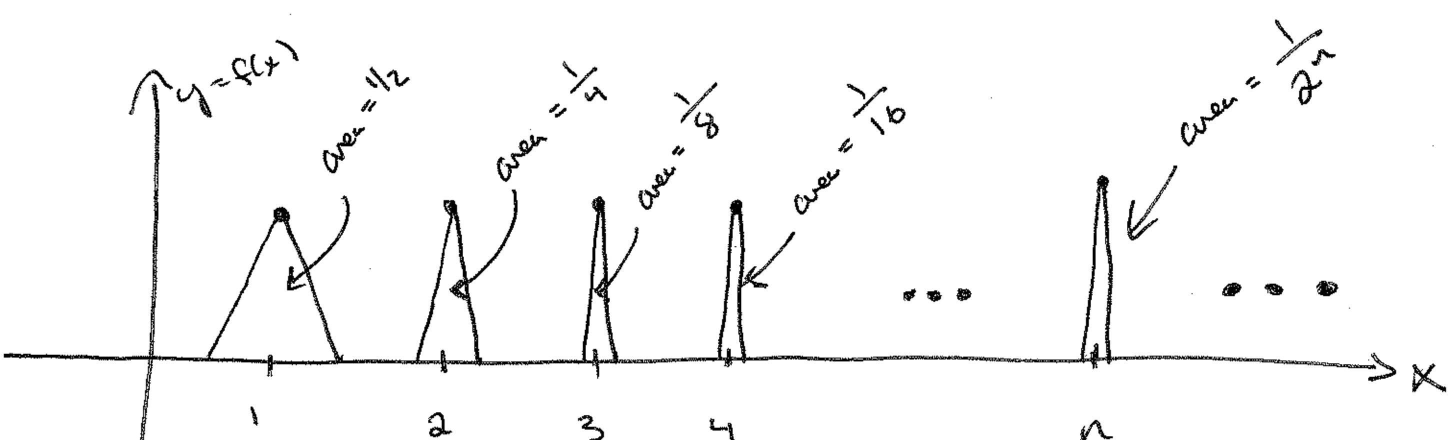
But $\int f(x)\,dx$ diverges.



(b) (5 points - all or nothing) If $\int_0^{\infty} f(x)\,dx$ converges, does the series $\sum_{n=0}^{\infty} a_n$ converge?

No. Integral test requires f to be decreasing, so let's break that!

Consider $a_n = 1$ for all n. Then $\sum a_n$ diverges.



Then, $\int^{\infty} f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{2} = \frac{1/2}{1 - 1/2} = 1$ converges.

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