Exam 2

Math 133
October 16^{th}, 2012

Name: Key

Question	Points	Your Score
Q1	15	
Q2	15	
Q3	15	
Q4	20	
Q5	17	
Q6	18	
TOTAL	100	

Read all of the following information before starting the exam:

- Show all work, clearly and in order. "Answers" without justification will receive zero credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: `https://www.msu.edu/~ombud/academic-integrity/index.html`. Simply put, don't cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!

Problem 1 (15 points) Evaluate each of the following integrals:

(a)

Note: $7^x = e^{x \ln 7}$.

$$\int_0^2 7^x\, dx = \int_0^2 e^{(\ln 7)\cdot x}\, dx = \frac{1}{\ln 7} e^{(\ln 7)\cdot x}\Big|_0^2$$

$$= \frac{1}{\ln 7}\cdot 7^x\Big|_0^2 = \frac{49-1}{\ln 7} = \boxed{\frac{48}{\ln 7}}$$

(b)

$$\int_1^e 3x\ln(5x)\, dx = \int_1^e 3x\ln 5 + 3x\ln x\, dx.$$

$$= \frac{3}{2}x^2\cdot\ln 5\Big|_1^e + 3\int_1^e x\ln x\, dx$$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

$$= \frac{3}{2}\ln 5\,(e^2-1) + 3\left(\frac{1}{2}x^2\ln x\Big|_1^e - \int_1^e \frac{1}{2}x\, dx\right)$$

$$= \frac{3}{2}\ln 5\,(e^2-1) + 3\left(\frac{e^2}{2} - 0 - \frac{1}{2}\frac{x^2}{2}\Big|_1^e\right)$$

$$= \frac{3}{2}\ln 5\,(e^2-1) + \frac{3}{2}e^2 - \frac{3}{4}(e^2-1).$$

Problem 2 (15 points)

(a) Find $\frac{dy}{dx}$, where $y = (2x+5)^{\sin(x)}$.

$$\ln y = \sin x \cdot \ln(2x+5).$$

$$\frac{1}{y} y' = \cos x \ln(2x+5) + \sin x \cdot \frac{2}{2x+5}$$

$$\frac{dy}{dx} = (2x+5)^{\sin x} \left[\cos x \ln(2x+5) + \frac{2}{2x+5} \cdot \sin x \right].$$

(b) Find $\frac{dy}{dx}$, where $y = \sinh(2\ln(x))$.

$$\frac{dy}{dx} = \cosh(2\ln x) \cdot \frac{2}{x}$$

Problem 3 (15 points) Evaluate the following integral:

$$I = \int \sin(x)\cos(3x)dx$$

$u = \cos 3x \qquad dv = \sin x \, dx$

$du = -3\sin 3x \, dx \qquad v = -\cos x$

$$I = -\cos x \cos 3x - 3 \int \cos x \sin 3x \, dx$$

~~u = cos~~ $\quad u = \sin 3x \qquad dv = \cos x \, dx$

$du = 3\cos 3x \, dx \qquad v = \sin x$

$$= -\cos x \cos 3x - 3\left(\sin x \sin 3x - 3 \underbrace{\int \sin x \cos 3x \, dx}_{= I} \right)$$

$$= -\cos x \cos 3x - 3 \sin x \sin 3x + 9I$$

$$\Rightarrow \quad -8I = -\cos x \cos 3x - 3 \sin x \sin 3x + C.$$

$$I = \frac{1}{8}\left(\cos x \cos 3x + 3 \sin x \sin 3x \right) + C.$$

Problem 4 [Zombies Attack Well's Hall!] (20 points) Let $y(t)$ be the number of zombies in Well's Hall. At time $t = 0$, a bizarre accident involving a Calculus midterm turned exactly 1 human into a zombie, so assume $y(0) = 1$.

The growth model $\frac{dy}{dt} = ky$, $k > 0$ admits unbounded growth, which is is obviously incorrect. Instead, assume

$$\frac{dy}{dt} = ky - \frac{ky^2}{100}.$$

The $-ky^2/100$ term accounts for zombies having a more difficult time finding humans at longer time. Solve for $y(t)$ in this model, and for simplicity, assume $k = 2$.

$$\frac{dy}{dt} = 2y - \frac{2y^2}{100} = -\frac{1}{50}\left(-100y + y^2\right)$$

$$\int \frac{1}{y(y-100)}\,dy = \int -\frac{1}{50}\,dt = -\frac{1}{50}t + C_0.$$

$$\frac{1}{y(y-100)} = \frac{A}{y} + \frac{B}{y-100} \qquad 1 = A(y-100) + By$$

$$= \frac{1}{100}\left[\frac{-1}{y} + \frac{1}{y-100}\right] \qquad \begin{array}{ll} y=100: & 1 = 100B \\ y=0: & 1 = -100A \end{array}$$

$$\int \frac{1}{y(y-100)}\,dy = \frac{1}{100}\left(\ln|y-100| - \ln|y|\right) = \frac{1}{100}\ln\left|\frac{y-100}{y}\right|$$

$$\Rightarrow \quad \frac{1}{100}\ln\left|\frac{y-100}{y}\right| = -\frac{1}{50}t + C_0$$

$$\ln\left|\frac{y-100}{y}\right| = -2t + C_1$$

$$\frac{y-100}{y} = Ae^{-2t}$$

$y(0)=1$:

$$-99 = A.$$

$$y - 100 = -99y\,e^{-2t}$$

$$y\left(1 + 99e^{-2t}\right) = 100$$

$$y = \frac{100}{1 + 99e^{-2t}}$$

Problem 5 (17 points) Integrate the following problem: (C.F. #104, Chap 8 Add Exercises)

$$I = \int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{1+y} \cdot 2y\,dy.$$

$y = \sqrt{x}$

$dy = \frac{1}{2} x^{-1/2} dx$

$2\sqrt{x}\,dy = dx$

$2y\,dy = dx$

$u = \sqrt{1+y}$ $\quad dv = 2y\,dy$

$du = \frac{1}{2}(1+y)^{-1/2} dy$ $\quad v = y^2$

$$I = y^2\sqrt{1+y} - \int \frac{y^2}{2}(1+y)^{-1/2} dy$$

$$= y^2\sqrt{1+y} - \frac{1}{2}\int \frac{y^2}{(1+y)^{1/2}} dy.$$

$z = 1+y \iff y = z-1,\quad dy = dz$

$$I = y^2\sqrt{1+y} - \frac{1}{2}\int \frac{(z-1)^2}{z^{1/2}} dz$$

$$= y^2\sqrt{1+y} - \frac{1}{2}\int \frac{z^2 - 2z + 1}{z^{1/2}} dz$$

$$= x\sqrt{1+\sqrt{x}} - \frac{1}{2}\int z^{3/2} - 2z^{1/2} + z^{-1/2}\,dz$$

$$= x\sqrt{1+\sqrt{x}} - \frac{1}{2}\left(\frac{2}{5}z^{5/2} - 2\,\frac{2}{3}z^{3/2} + 2z^{1/2}\right) + C$$

$z = y+1 = \sqrt{x}+1.$

$$= x\sqrt{1+\sqrt{x}} - \frac{1}{2}\left(\frac{2}{5}(\sqrt{x}+1)^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + 2(\sqrt{x}+1)^{1/2}\right) + C$$

Problem 6 (18 points) Evaluate the following integral:

$$\int \frac{(-3+4x-x^2)^{3/2}}{(x-2)^6}\,dx$$

Can complete square, or use a u-sub.

$y = x-2 \qquad dy = dx.$

$y+2 = x.$

$$-3+4x-x^2 = -3+4(y+2)-(y+2)^2$$

$$= -3+4y+8-(y^2+4y+4)$$

$$= -3+\cancel{4y}+8-y^2-\cancel{4y}-4$$

$$= 1-y^2.$$

$$\int \frac{(1-y^2)^{3/2}}{y^6}\,dy = \int \frac{(\cos^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\,d\theta$$

$y = \sin\theta \implies 1-y^2 = \cos^2\theta$

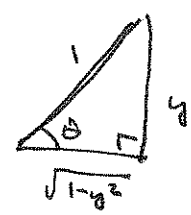
$dy = \cos\theta\,d\theta$

$$= \int \frac{\cos^4\theta}{\sin^6\theta}\,d\theta = \int \cancel{\tan^4\theta}\ \cot^4\theta\,\csc^2\theta\,d\theta$$

$u = \cot\theta \qquad du = -\csc^2\theta\,d\theta$

$$= -\int u^4\,du = -\frac{1}{5}u^5 + C$$

$$= -\frac{1}{5}\cot^5\theta + C.$$



$$= -\frac{1}{5}\left(\frac{\sqrt{1-y^2}}{y}\right)^5 + C$$

$$= -\frac{1}{5}\left(\frac{\sqrt{1-(x-2)^2}}{x-2}\right)^5 + C$$

Bonus: (5 points - all or nothing)
Suppose that $f : [0,1] \to \mathbb{R}$ is a smooth function that satisfies $\int_0^1 f(x)f''(x)\,dx = 0$. In addition, we know that $f(0) = f'(1) = 0$. Show that $f(x) \equiv 0$ for all $x \in [0,1]$.

$$0 = \int_0^1 \underbrace{f(x)}_{u} \underbrace{f''(x)\,dx}_{dv}$$

$$du = f'(x)\,dx \qquad v = f'(x)$$

$$= f(x) f'(x) \Big|_0^1 - \int_0^1 [f'(x)]^2\,dx$$

$$\Rightarrow \quad 0 = \int_0^1 (f'(x))^2\,dx$$

$$\Rightarrow \quad f'(x) = 0 \quad \text{all } x$$

$$\Rightarrow \quad f(x) = \text{Constant}$$

but $f(0) = 0$, so $f(x) = 0$.

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