# Exam 1

Math 133
September $18^{th}$, 2012

**Name:** Key

| Question | Points | Your Score |
|---|---|---|
| Q1 | 20 | |
| Q2 | 10 | |
| Q3 | 10 | |
| Q4 | 20 | |
| Q5 | 20 | |
| Q6 | 20 | |
| TOTAL | 100 | |

**Read all of the following information before starting the exam:**

- Show all work, clearly and in order. "Answers" without justification will receive zero credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- All exams at Michigan State University are governed by our Academic Integrity Policy: `https://www.msu.edu/~ombud/academic-integrity/index.html`. Simply put, don't cheat. There are serious consequences.
- Wait until instructed to begin exam to start. Good luck!

**Problem 1** (20 points) Compute $\frac{dy}{dx}$ for each of the following:

(a) $y = e^{-\pi x}$

$$\frac{dy}{dx} = e^{-\pi x} \frac{d}{dx}(-\pi x) = -\pi e^{-\pi x}$$

(b) $y = x^{-\pi}$

$$\frac{dy}{dx} = (-\pi)\, x^{-\pi-1} = -\pi\, x^{-(1+\pi)}$$

(c) $y = 2\ln\left(\frac{7}{x^2}\right)$ $= 2(\ln 7 - \ln x^2) = 2\ln 7 - 4\ln x$

$$\frac{dy}{dx} = \frac{-4}{x}.$$

(d) $y = \ln(2\sin(\ln x))$ $= \ln 2 + \ln(\sin \ln x)$

$$\frac{dy}{dx} = \frac{1}{\sin \ln x}(\sin \ln x)'$$

$$= \frac{1}{\sin(\ln x)} \cos(\ln x)\,(\ln x)'$$

$$= \frac{1}{\sin(\ln x)} \cos(\ln x)\, \frac{1}{x}.$$

**Problem 2** (10 points) The length of the curve which travels along the graph of $y = e^{5x} \ln x$ from the point $(1, 0)$ to the point $(e, e^{5e})$ is found by evaluating an appropriate integral. Set up, but *do not evaluate* this integral.

$$\frac{dy}{dx} = 5e^{5x} \ln x + \frac{e^{5x}}{x}.$$

$$L = \int_1^e \sqrt{1 + \left(5e^{5x}\ln x + \frac{e^{5x}}{x}\right)^2}\, dx$$

**Problem 3** (10 points)

(a) Given the function $f(x) = 3(x - 4)^2 + 1$ whose domain is $x \le 4$, find $f^{-1}(x)$.

$$y = 3(x-4)^2 + 1$$

$$y - 1 = 3(x-4)^2$$

$$\frac{y-1}{3} = (x-4)^2$$

$$\pm\sqrt{\frac{y-1}{3}} = x - 4$$

$$x = 4 \pm \sqrt{\frac{y-1}{3}}$$

we take the neg square root b/c $x \le 4$:

$$x = 4 - \sqrt{\frac{y-1}{3}}.$$

Swapping $x \leftrightarrow y$:

$$\boxed{f^{-1}(x) = 4 - \sqrt{\frac{y-1}{3}}}$$

(b) Compute $(f^{-1})'(4)$.

$b = 4$. Note $f(3) = 3(-1)^2 + 1 = 4$.

so $(f^{-1})'(4) = \dfrac{1}{f'(3)}$

$$f'(x) = 6(x-4)$$

$$f'(3) = -6$$

so $\boxed{(f^{-1})'(3) = -\dfrac{1}{6}}$

**Problem 4** (20 points) A force of 3 lbs is required to stretch an ideal spring 1 ft from its resting, equilibrium position.

(a) Compute the spring constant, $k$. What units, if any, does $k$ have?

$$F = kx$$

$$3 = k(1) \quad \text{so} \quad \boxed{k = 3 \ \frac{\text{lbs}}{\text{ft}}}$$

(b) How much work is required to stretch the spring 2 feet from its equilibrium position?

$$W = \int F\,dx = \int_0^2 3x\,dx = \frac{3}{2} x^2 \Big|_0^2$$

$$= \frac{3}{2}(2)^2 = \boxed{6 \ \text{ft lbs}}$$

**Problem 5** (20 points) Evaluate each of the following integrals:

(a)

$$\int_{\ln(1/2)}^{\ln(2)} e^x \sin(2e^x - 1)\, dx$$

$u = 2e^x - 1$

$du = 2e^x dx, \quad \frac{du}{2e^x} = dx$

$= \int \cancel{e^x} \sin(u) \frac{du}{2\cancel{e^x}} = \frac{1}{2}\int \sin u \, du$

$= -\frac{1}{2} \cos u \Big|_{x=\ln(1/2)}^{x=\ln 2} = -\frac{1}{2} \cos u \Big|_{u=0}^{u=3}$

$x = \ln(1/2)$:

$u = 2e^{\ln(1/2)} - 1 = 1 - 1 = 0$

$= \boxed{-\frac{1}{2}(\cos 3 - 1)}$

$x = \ln 2$:

$u = 2e^{\ln 2} - 1 = 4 - 1 = 3$

(b)

$$\int \frac{3x^2 e^{x^3}}{2 + e^{x^3}} dx$$

$u = 2 + e^{x^3} \quad du = 3x^2 e^{x^3} dx$

$= \int \frac{du}{u} = \ln|u| + C$

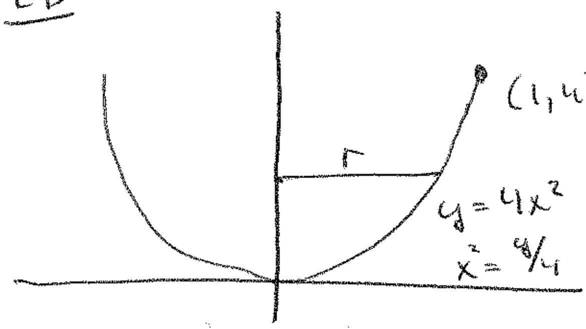
$= \boxed{\ln(2 + e^{x^3}) + C}$

can remove $|\cdot|$ signs because $2 + e^{x^3} \geq 0$.

**Problem 6** (20 points) The region in the first quadrant bounded below by $y = 4x^2$ and above by $y = 4$ is revolved about the $y$-axis.

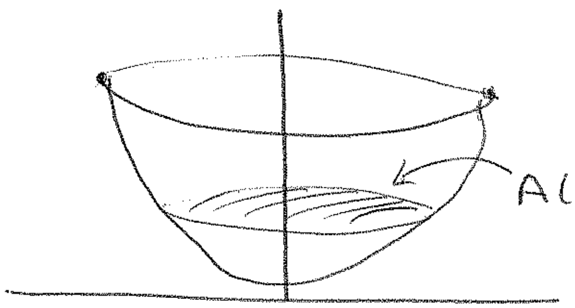
(a) Find the volume of the resulting solid.

2D



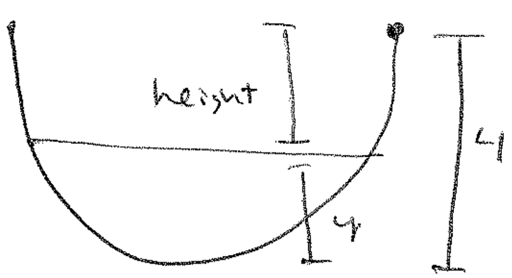

$$V = \int \pi r^2 \, dy = \pi \int_0^4 \frac{y}{4}\, dy = \frac{\pi}{4} \frac{y^2}{2}\Big|_0^4 = \frac{\pi}{4} \cdot \frac{16}{2} = \boxed{2\pi}$$

3D



$A(y) = \pi r^2 = \pi \left(y/4\right)$

(b) Now, suppose that the solid represents the interior of a tank, and this tank is filled with water with constant density $\rho = 15 N/m^3$. Set up, but *do not evaluate* an integral that describes the amount of work necessary to pump all of the water to the surface of the tank at $y = 4$.




$$W = \int \rho \cdot A \cdot (\text{height})\, dy = \boxed{15 \int_0^4 \left(\pi \frac{y}{4}\right)(4-y)\, dy}$$

**Bonus:** (5 points - all or nothing)
Show that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x^{11} + 2x^5 + 2x$ has an inverse.

$$f' = \underbrace{33x^{10}}_{\geq 0} + \underbrace{10x^4}_{\geq 0} + 2 \geq 2.$$

$\therefore$ f is always increasing

$\therefore$ f has an inverse.

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