CHAPTER 10 NOTES

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1. Two Point Boundary Value Problems

All of the problems listed in $14-20$ ask you to find eigenfunctions for the problem

$$y'' + \lambda y = 0 \tag{1}$$

with some prescribed data on the boundary. To solve this, you always have to deal with three cases.

Case I: $\lambda = -\mu^2 > 0$. The problem is then

$$y'' - \mu^2 y = 0$$

whose characteristic equation is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$. The solution is then

$$y = a_1 e^{\mu x} + a_2 e^{\mu x}.$$

For the purposes of evaluating this function at two different points, it's convenient to express this in terms of hyperbolic sines and cosines. The definitions of these functions are

$$\cosh(x) := \frac{e^x + e^{-x}}{2}, \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

from which we can solve for the e^x and e^{-x} functions:

$$e^x = \cosh(x) + \sinh(x), \quad e^{-x} = \cosh(x) - \sinh(x).$$

If you substitute this into our equation for $y(x)$, we can exchange the constants a_1 and a_2 for new constants c_1 and c_2 and write the solution as

$$y(x) = c_1 \cosh(x) + c_2 \sinh(x).$$

There are a couple of very useful facts about the hyperbolic sine and cosine functions. 1.) Since e^x, e^{-x} are always positive, we know $\cosh(x) \neq 0$ for *any* x. 2.) Since $\frac{d}{dx}\sinh(x) = \cosh(x) > 0$, we know that $\sinh(x)$ is an increasing function. Furthermore, by inspection we know that $\sinh(0) = 0$. Therefore the *only* zero of sinh is at $x = 0$.

Case II: $\lambda = 0$. This is the easy case. The differential equation is

$$y'' = 0$$

whose solution is (after integrating twice) $y = c_1 + c_2 x$. You can also get this from looking at the characteristic equation: $r^2 = 0$ whose roots are $r = 0$ with multiplicity two. The solution is apparently $y = c_1 e^{0x} + c_2 x e^{0x}$.

Case III: $\lambda = \mu^2 > 0$ The differential equation is

$$y'' + \mu^2 y = 0$$

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whose characteristic equation is $r = \pm \mu i$. Hence the solution is

$$y = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Sine and cosine have plenty of zeros to work with. This is usually the 'good' case.

The three cases I worked out for you are given in formulas (21), (26) and (28) in your textbook.

Problem #14 Find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all the eigenvalues are real.

$$y'' + \lambda y = 0, \quad y(0) = y'(\pi) = 0.$$

We need to break this into cases.

Case I: $\lambda = -\mu^2 < 0$. The solution to this is $y(x) = c_1 \cosh(x) + c_2 \sinh(x)$. Plugging in $y(0) = 0$, we see that $c_1 = 0$. Hence $y = c_2 \sinh(\mu x)$ and $y' = c_2 \mu \cosh(\mu x)$ with $y'(\pi) = c_2 \mu \cosh(\mu \pi) = 0$. Since $\mu \neq 0$ and $\cosh(\mu\pi) \neq 0$ (cosh has NO zeros!) we must force $c_2 = 0$. Hence $y \equiv 0$ is the only solution out of this case.

Case II: $\lambda = 0$ with solution $y = c_1 + c_2 x$. Plugging in 0 and π we see that $c_1 = c_2 = 0$. We're not making much progress!

Case III: $\lambda = \mu^2 > 0$ with solution $c_1 \cos(\mu x) + c_2 \sin(\mu x)$. Plugging in $x = 0$ we see that $c_1 = 0$ hence $y = c_2 \sin(\mu x)$. Differentiating this we have $y' = c_2 \mu \cos(\mu x)$. Plugging in π we have $y'(\pi) = 0 = c_2 \mu \cos(\mu \pi)$. Now we get non-trivial solutions! We must have $\mu\pi$ be a zero of cosine which means $\mu \pi = \pi/2, 3\pi/2, 5\pi/2, \ldots$ and hence $\mu = (\text{odd integer})/2 = (2n-1)/2$ for some $n \geq 1$. Plugging this back into the original y, we have eigenfunction solutions $y_n(x) = \sin((2n-1)/2x)$ with associated eigenvalue $\lambda_n = \mu_n^2 = ((2n-1)/2)^2$.

2. Fourier Series

The functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ form an orthogonal set on the interval $[-L, L]$. That is,

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

This fact allows us to solve for the Fourier cofficients of a function. If we can express a function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2}$$

then the coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx; \tag{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \quad n \geq 1; \tag{4}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad n \geq 1. \tag{5}$$

One can derive these by multiplying equation (2) by $\cos(m\pi x/L)$ then integrating over the full period. Know these formulas.

3. The Fourier Convergence Theorem

The previous section stated that if we want to express a known function $f(x)$ as an infinite sum of sines and cosines, then the coefficients must be given by equations (3) - (5). The precise mathematical statement for when you can do this is given by theorem 10.3.1.

If $f(x)$ is periodic of period $2L$ and piecewise C^1 (that is f has a derivative that's continuous everywhere except at finitely many points) then f has a fourier series given by (2) and at points where f is discontinuous the series converges to the average left and right hand side values.

As an example, find the fourier series for

$$f(x) = \begin{cases} x^3, & 0 \leq x < 1, \\ 0, & -1 \leq x < 0, \end{cases}$$

where f is 2-periodic.

Since the period of f is two, then $L = 2/2 = 1$. Hence the constant term is given by

$$a_0 = \int_{-1}^{1} f(x)\ dx = \int_0^1 f(x)\ dx = \int_0^1 x^3\ dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}.$$

The cosine terms are given by

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\ dx = \int_0^1 f(x)\cos(n\pi x)\ dx = \int_0^1 x^3\cos(n\pi x)\ dx$$

and the sine terms are given by

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\ dx = \int_0^1 f(x)\sin(n\pi x)\ dx = \int_0^1 x^3\sin(n\pi x)\ dx$$

You can compute these integrals using tabular integration.

4. Even and Odd Functions

An even function is a function with the property that $f(-x) = f(x)$ whenever x is in the domain of f. An odd function is a function with the property that $f(-x) = -f(x)$ whenever x is in the domain of f. Note that is only makes sense to talk about whether or not a function is even or odd if the functions domain is symmetric about the origin. For example,

$$f(x) = x^3, \quad -\infty < x < \infty$$

is an odd function but

$$f(x) = x^3, \quad -1 < x < \infty$$

is not an odd function. Because $f(10)$ is defined, but $f(-10)$ is not defined. The fourier coefficients for even/odd functions only include half the terms because of properties of integration.

If $f(x)$ is an *odd* function, then

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\ dx = 0; \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx = 0, \quad n \geq 1; \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad n \geq 1. \end{aligned}$$

These are true since f and $f(x)\cos\left(\frac{n\pi x}{L}\right)$ are both odd functions. We can double the value of the integral and integrate over only half the interval for the third integral because $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is an even function. Hence for odd functions, we can express them as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad n \geq 1. \tag{6}$$

This result comes directly from equations (3) - (5) and a simple property about integrals.

Likewise, if $f(x)$ is an *even* function, then

$$f(x) = \frac{a_0}{2}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{0}^{L} f(x)\ dx \\ a_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \quad n \geq 1. \end{aligned} \tag{7}$$

Note that both of these integrals *only* depend on the value of $f(x)$ on the interval $[0, L]$.

4.1. **Extending a Function - Sine and Cosine Series.** Suppose we have a function $f(x)$ defined on an interval $[0, L]$ and we seek some representation of this function using sines and cosines. All of our formulas for the Fourier series depend of a function that's defined everywhere and is $2L$-periodic. In order to do this we need to extend $f(x)$ to the full line in such a way that it's $2L$-periodic. For starters we need to extend f to $(-L, L]$ and once we do that, we can declare f to be $2L$-periodic and the use the formulas (3) - (5) to compute the Fourier series for $f(x)$.

There are infinitely many ways we can extend the function to the full interval, but at least two of these are incredibly useful. If we choose to extend f so that it's an even function, then the Fourier series will only have cosine terms and likewise, if we choose to extend f so that it's an odd function, then the Fourier series will only have sine terms. What's remarkable is the fact that doing this, we get two different representations for the function on $[0, L]$ that look completely different but the infinite sum still adds to the *same* value! I.e. we have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad x \in (0, L)$$

and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = -\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad x \in (-L, 0)$$

if we choose to set the a_n's coming from the even extension of f and the b_n's coming from the odd extension of f given by (6) and (7)

5. Separation of Variables; Heat Conduction in a Rod

Please read pages 612-616 from your textbook.

The important equations are given by (17) which is

$$u_n(x,t) = e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

is a solution to

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(0,t) &= u(L,t) = 0 \quad t > 0. \end{aligned}$$

Thus the 'general' solution is given by linear combinations of these solutions (because the problem is linear) which is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \sin\left(\frac{n\pi}{L} x\right), \tag{8}$$

for some unknown constants c_n. (This is equation (19) in your textbook). In order to figure out what the constants of integration (c_n's) are, we need to know an initial temperature distribution on the rod given by

$$u(x,0) = f(x) \quad x \in [0, L]$$

However, plugging in $t = 0$ into equation (8) gives us

$$u(x,0) = f(x) \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right).$$

This is precisely the odd Fourier series for $f(x)$!! Hence is we set

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1,$$

then this solves the full problem.

6. Other Heat Conduction Problems

7. The Wave Equation: Vibrations of an Elastic String

The wave equation is given by

$$u_{tt} = a^2 u_{xx} \quad 0 < x < L, t > 0 \tag{9}$$
$$u(0,t) = u(L,t) = 0, \quad t > 0 \tag{10}$$
$$u(x,0) = f(x), \quad 0 < x < L \tag{11}$$
$$u_t(x,0) = g(x), \quad 0 < x < L, \tag{12}$$

where $u(x,t)$ is the displacement of the string from equilibrium at a given point x and time $t > 0$. Equation (10) means that the endpoints of the string are fixed. Equation (11) is an initial displacement given to the string and equation (12) is an initial velocity given to the string. We actually split this problem up into two separate problems:

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \quad 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 < x < L \\ u_t(x,0) &= 0, \quad 0 < x < L \end{aligned} \tag{13}$$

and

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \quad 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0, \quad t > 0 \\ u(x,0) &= 0, \quad 0, < x < L \\ u_t(x,0) &= g(x), \quad 0, < x < L. \end{aligned} \tag{14}$$

Since if we can find a solution u_1 that satisfies (13) and a solution u_2 that satisfies (14), then $u = u_1 + u_2$ satisfies equations (9) - (12).

If we use separation of variables $u(x,t) = X(x)T(t)$ on equation (9), we end up with

$$XT'' = a^2 X'' T$$

and after dividing by $a^2 XT$, we get

$$\frac{T''}{a^2 T} = \frac{X''}{X} \equiv -\text{const}.$$

This gives us two ODES:

$$T'' + a^2 \cdot \text{const} \cdot T = 0$$
$$X'' + \text{const} \cdot X = 0$$

whose solutions depend on the sign of the constant. If use the fact that $u(0,t) = u(L,t) = 0$, we must have $X(0) = X(L) = 0$ and so we must force this constant to be a positive constant. Hence the ODEs are actually

$$T'' + a^2\lambda^2 \cdot T = 0,$$
$$X'' + \lambda^2 X = 0,$$

whose solutions are

$$\begin{aligned} T(t) &= c_1 \cos(\lambda a t) + c_2 \sin(\lambda a t) \\ X(x) &= c_3 \cos(\lambda x) + c_3 \sin(\lambda x) \end{aligned}$$

Again, since $X(0) = X(L) = 0$ we need $c_3 = 0$ and λL to be a zero of sin. Hence $\lambda L = n\pi$, or written another way, $\lambda = \lambda_n = \frac{n\pi}{L}$. The spatial part is given by

$$X(x) = \sin(\lambda x). \tag{15}$$

Now we can solve for $T(t)$. This gives

$$T(t) = c_1 \cos\left(\frac{n\pi}{L} at\right) + c_2 \sin\left(\frac{n\pi}{L} at\right). \tag{16}$$

For problem (13), we need $u_t(x, 0) = T'(0)X(0) = 0$, and hence we should force $c_2 = 0$. This gives a solution,

$$u_n(x,t) = \cos\left(\frac{n\pi}{L} at\right) \sin\left(\frac{n\pi}{L} x\right)$$

from which we can take linear combinations to give a solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L} at\right) \sin\left(\frac{n\pi}{L} x\right)$$

which solves equations (9), (10) and (12). The only remaining piece to solve is equation (11) which means we need to choose the coefficients to satisfy

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right).$$

This is *precisely* the odd-Fourier series (sine series) for the function $f(x)$. Hence we need to set

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx. \tag{17}$$

and we have a solution to problem (13).

In order to account for zero initial displacement but with initial velocity, we can back up to equations (15) and (16). Since we want $u(x, 0) = 0$, we should force $c_1 = 0$ and we are free to choose c_2. If we take a time derivative of u and plug in $t = 0$, we get

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{L} a \sin\left(\frac{n\pi}{L} x\right).$$

If we take the odd Fourier series for $g(x)$, with coefficients given by

$$b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) \, dx. \tag{18}$$

we see we have the equation

$$\sum_{n=1}^{\infty} c_n \frac{n\pi}{L} a \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right).$$

Forcing equality for each of the coefficients gives

$$c_n = \frac{L}{n\pi a} b_n = \frac{L}{n\pi a} \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) \, dx.$$

This solves every piece of equation (14) which is the zero displacement with non-zero initial velocity.