PERIODIC HOMOGENIZATION FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this note, we prove the periodic homogenization for a family of nonlinear nonlocal "elliptic" equations with oscillatory coefficients. Such equations include, but are not limited to Bellman equations for the control of pure jump processes and the Isaacs equations for differential games of pure jump processes. The existence of an effective equation and convergence the solutions of the family of the original equations is obtained. An inf-sup formula for the effective equation is also provided.

1. INTRODUCTION

In this note, we consider the homogenization of the family of nonlinear, nonlocal (integro-differential) equations given by

$$\begin{cases} F(u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } D \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \setminus D, \end{cases} \tag{1.1}$$

where D is an open, bounded domain in $\mathbb{R}^n$. In this context the operator, F, will take the form:

$$F(u, \frac{x}{\varepsilon}) = \inf_\alpha \sup_\beta \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) \mathrm{dy} \right\}.$$

Such operators appear in Bellman-Isaacs equations related to optimal control and to two player games involving pure jump Lévy processes. We assume that the family of operators, $F(u, x)$, are $\mathbb{Z}^n$ periodic with respect to their dependence on $\mathbb{R}^n$. These periodicity assumptions, the exact assumptions on F, and the precise meaning of equations such as (1.1) will be elaborated below.

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Recently much attention has been paid to modeling with jump diffusion processes, particularly in financial mathematics and engineering ([1], [8], [13], [14], [23], [24] and many more). Here we investigate the macroscopic contribution of order 1 oscillations of the equation at a microscopic spacial scale, modeled by the equations dependence on x/ε. The expectation is that due to the periodic nature of oscillations, the microscopic behavior can be "averaged" out to produce a simpler model approximating the behavior at the macroscopic level (homogenization occurs). Our current study is restricted to equations involving the generators for pure jump processes, which represents a vital first step towards understanding the more general behavior of generators of jump-diffusions.

1.1. **Main Theorem and Assumptions.** The results we prove will show the existence of an effective nonlocal equation such that the family of solutions governed by (1.1) converges locally uniformly to the solution of this effective equation. The important features are that the effective equation is nonlocal, elliptic, and translation invariant, given by

$$\begin{cases} \bar{F}(\bar{u},x)=0 & \text{in } D \\ \bar{u}=g & \text{on } D^c. \end{cases} \tag{1.2}$$

This behavior of u^ε is described in the main theorem of the note:

Theorem 1.1. *Assume F1, F2a, F2b,F3, F4 listed below and that the comparison theorem holds for (1.1). Then there exists a translation invariant operator, $\bar{F}$, which describes a nonlocal "elliptic" equation such that for any choice of uniformly continuous data, g, the solutions of (1.1) converge locally uniformly to a unique $\bar{u}$, and $\bar{u}$ solves (1.2). Moreover $\bar{F}$ is "elliptic" with respect to the same extremal operators as the original operator, F.*

Remark 1.2. We say that the nonlocal elliptic operator, $\bar{F}$, is translation invariant if $\bar{F}(u,x+y)=\bar{F}(u(\cdot+y),x)$ whenever u is such that $\bar{F}(u,x+y)$ is well defined. The notions of "ellipticity" and extremal operators for operators such as F are discussed in the Appendix (Section, 7).

The context, definitions, and properties of equations such as (1.1) and (1.2) are taken from [4], [5], [9], and [10]. We point out the very important difference in sign convention regarding sub and super solutions of (1.1). In this note, we adopt the convention from [10], namely that a subsolution will solve $F(u,x)\geq 0$.

As to be expected, the nonlocal nature of (1.1) introduces some additional difficulties that are not present in its second order elliptic counterpart. One such difficulty is that the effective equation is defined on a space of appropriate smooth functions on $\mathbb{R}^n$, instead of the space of symmetric matrices (which can be associated to the paraboloids they give rise to as functions). The second major difficulty is that the natural scaling of the problem is given by the transformation $u(x)$ maps to $\varepsilon^\sigma u(x/\varepsilon)$, for $\sigma < 2$ (this is elaborated below). However, the class of functions for the definition of the effection equation is not invariant under this scaling! In the second order case, the relevant scaling is $u(x)$ maps to $\varepsilon^2 u(x/\varepsilon)$, and the paraboloids are indeed invariant with respect to this transformation. In previous works, this translation invariance was fundamentally used to identify the effective equation, [12]. We circumvent this difficulty by appropriately modifying the definition of the "corrector" equation and modifying the known proofs of the existence of the effective operator to lessen the strategic use of paraboloids in previous works (this is described in Section 2).

We make some additional assumptions about the operator, F. First, we require for the convenience of concrete notation that

- (F1) (Inf-Sup Form of The Operator, F)

$$F(\phi,x) = \\ \inf_\alpha \sup_\beta \big\{ f^{\alpha\beta}(x) + \int_{\mathbb{R}^n} (\phi(x+y)+\phi(x-y)-2\phi(x)) K^{\alpha\beta}(x,y)\mathrm{dy} \big\}. \tag{1.3}$$

This assumption does not appear to be absolutely necessary for the results proved in this note, but it does give a very general, concrete form of equations with which to work. When we write (1.1) for a smooth and appropriately integrable ϕ, we actually mean the expression given by

$$F(\phi,\frac{x}{\varepsilon}) = \\ \inf_\alpha \sup_\beta \big\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (\phi(x+y)+\phi(x-y)-2\phi(x)) K^{\alpha\beta}(\frac{x}{\varepsilon},y)\mathrm{dy} \big\}. \tag{1.4}$$

This notation is meant to be in agreement with the same conventions for local differential equations. For example it is now standard that if we define the operator $L(D^2u,x) = a_{ij}(x)u_{x_ix_j}(x)$, then by writing the expression $L(D^2u,x/\varepsilon)$, we actually mean $a_{ij}(x/\varepsilon)u_{x_ix_j}(x)$.

For purposes of regularity of the solutions of (1.1), we will require that each integration kernel, $K^{\alpha\beta}$, is symmetric in the variable of integration, that all the kernels in the family satisfy the same scaling with respect to y and that they are "uniformly elliptic":

- (F2a) (Symmetry of The Kernels)

$$K^{\alpha\beta}(x,-y) = K^{\alpha\beta}(x,y), \tag{1.5}$$

- (F2b) (Scaling Propertry of The Kernels) For some $0 < \sigma < 2$,

$$K^{\alpha\beta}(x,\lambda y) = \lambda^{-n-\sigma} K^{\alpha\beta}(x,y), \tag{1.6}$$

- (F3) (Uniform Ellipticity) There exist positive constants, $\lambda < \Lambda$ such that

$$\frac{\lambda}{|y|^{n+\sigma}} \leq K^{\alpha\beta}(x,y) \leq \frac{\Lambda}{|y|^{n+\sigma}}. \tag{1.7}$$

We say σ in (1.6) and (1.7) is the order of the operators corresponding to these kernels.

Typically, the operators in (1.3) naturally arise as the generators corresponding to a pure jump process on $\mathbb{R}^n$ and are commonly written as

$$\int_{\mathbb{R}^n} (\phi(x+y) - \phi(x) - [D\phi(x)\cdot y]\mathbb{1}_{|y|\leq 1}) K^{\alpha\beta}(x,y)\mathrm{dy}.$$

However, as utilized in [10] the symmetry property allows us to write (1.3) using the symmetric difference of ϕ instead of the gradient of ϕ. The scaling assumption, (1.6), tells us that if v satisfies $F(v,x) = 0$ in a set, $\varepsilon^{-1}D$, then $u(x) = \varepsilon^{\sigma} v(x/\varepsilon)$ satisfies $F(u, x/\varepsilon) = 0$ in the set, D. This will be an indispensable property for identifying the effective equation.

Finally we have

- (F4) (Periodicity of F)

$$F(u, x+z) = F(u(\cdot + z), x) \quad \text{for all} \quad z \in \mathbb{Z}^n,$$

which is satisfied whenever $f^{\alpha\beta}$ and $K^{\alpha\beta}(\cdot, y)$, for y fixed, are $\mathbb{Z}^n$-periodic.

1.2. **Comments on Uniqueness For (1.1).** In this work, the analysis of the Homogenization for (1.1) is in fact completely unrelated to the assumptions of existence and uniqueness for (1.1) – the only thing that matters is whether or not the comparison theorem holds. It is for this reason that we do not give the most precise assumptions for existence and uniqueness (e.g. a comparison theorem) for such equations. The most current results for existence and uniqueness can be found in [5]. The difficulty lies in the fact that the uniqueness theory is only well developed for operators which take the form

$$\int_{\mathbb{R}^n} (\phi(x + j^{\alpha\beta}(x,y)) + \phi(x - j^{\alpha\beta}(x,y)) - 2\phi(x)) \, |y|^{-n-\sigma} \, \mathrm{dy},$$

but not as much for the form used in this work,

$$\int_{\mathbb{R}^n} (\phi(x+y) + \phi(x-y) - 2\phi(x)) K^{\alpha\beta}(x,y)\mathrm{dy}.$$

Both forms appear to be used significantly in the literature, as evidenced in [5] and [22] for the former and [10] and [6] for the latter, and the many references contained within each.

One particular assumption which gives uniqueness of (1.1) and also respects F2a, F2b, and F3 is

- (F5) (Uniqueness) $f^{\alpha\beta}$ are uniformly bounded and uniformly continuous, and the nonlocal operators in (1.3) are given as
$$\int_{\mathbb{R}^n} (\phi(x + j^{\alpha\beta}(x, y)) + \phi(x - j^{\alpha\beta}(x, y)) - 2\phi(x)) \, |y|^{-n-\sigma} \, \mathrm{dy},$$
with
$$j^{\alpha\beta}(x, y) = c^{\alpha\beta}\big(x, \frac{y}{|y|}\big) y, \tag{1.8}$$
and $c^{\alpha\beta}$ are Lipschitz in x, uniformly in y, α, and β.

It can be checked by a change of variables, $w = j(x, y)$, that these particular forms of j will result in an operator of the form
$$\int_{\mathbb{R}^n} (\phi(x + y) + \phi(x - y) - 2\phi(x)) K^{\alpha\beta}(x, y) \mathrm{dy}.$$
More details regarding (F5) and uniqueness appear in the appendix.

2. Background and Main Ideas

The main feature of equations such as (1.1) which will allow us to identify the effective equation, $\bar{F}$, is the behavior of solutions to a certain family of obstacle problems. This method was introduced in [12], and our strategy very closely follows the one presented there. In this note, the methods of [12] have been adapted to suit the nonlocal nature of (1.1).

2.1. **Motivation of The Corrector Equation.** To motivate the present investigation, we recall some key ideas used for homogenization in previous works: ([7], Chapter 1 - Section 2), ([12], Section 1), and ([18], Sections 2 and 3), but presented in the context of nonlocal equations. We proceed with the linear case for the sake of clear presentation:
$$F(\phi, \frac{x}{\varepsilon}) = f(\frac{x}{\varepsilon}) + [L^{\varepsilon}\phi](x) = f(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (\phi(x + y) + \phi(x - y) - 2\phi(x)) K(\frac{x}{\varepsilon}, y) \mathrm{dy}.$$
The two main observations are:

1. u^{ε} solving (1.1) should (formally!) obey an expansion as
$$u^{\varepsilon}(x) = \bar{u}(x) + \varepsilon^{\sigma} v(\frac{x}{\varepsilon}) + o(\varepsilon^{\sigma})$$

2. L^ε actually has two separate scales, one is the location of the centered difference given by x and the other is the location of the kernel evaluation given by x/ε. As stated in the Appendix– Lemma 7.5, if we define

$$[L\phi(z)](x) = \int_{\mathbb{R}^n} (\phi(z+y) + \phi(z-y) - 2\phi(z))K(x,y)\mathrm{dy}, \tag{2.1}$$

then for x fixed, $[L\phi(z)](x)$ is actually a uniformly continuous function of z (the location of the centered difference), depending only on $\|D^2\phi\|_\infty$ and the ellipticity constants for L. In particular the uniform continuity is completely independent of x!

To simplify matters even further, for the sake of presentation, let us temporarily assume the most basic form of the equation– L is given by the fractional Laplacian:

$$[L\phi(z)](x) = \int_{\mathbb{R}^n} (\phi(z+y) + \phi(z-y) - 2\phi(z))\,|y|^{-n-\sigma}\,\mathrm{dy}.$$

Our original equation, (1.1), now reads

$$Lu^\varepsilon(x) = f(\frac{x}{\varepsilon}),$$

for some f.

We would now like to make sense of (1.1) while plugging in the expansion of u^ε for all ε. Doing so would require

$$L\bar{u}(x) + Lv(\frac{x}{\varepsilon}) = f(\frac{x}{\varepsilon}). \tag{2.2}$$

Here we have used F2a for $L(\varepsilon^\sigma v(\cdot/\varepsilon))(x) = Lv(x/\varepsilon)$. Furthermore, as $\varepsilon \to 0$, x/ε becomes a global variable and x can be considered fixed, hence we are more or less looking for periodic solutions of

$$L\bar{u}(x) + Lv(y) = f(y) \quad \text{for} \quad y \in \mathbb{R}^n.$$

Thanks to the self adjointness of L as the fractional Laplacian, and the fact that the only bounded global solutions of $L = 0$ are constants, the Fredholm Alternative tells us that the previous line will only have a solution when

$$\int_Q f(y)\mathrm{dy} - L\bar{u}(x) = 0,$$

which will only happen in very special occurrences of $L\bar{u}(x)$.

The only reasonable way to legitimize the expansion and salvage the information from (1.1) for all ε would be to force this integral to be zero, depending only on $\bar{u}$ and x. In this case we take the constant, $\bar{F}(\bar{u}, x)$, such that

$$\bar{F}(\bar{u}, x) + \int_Q f(y)\mathrm{d}y - L\bar{u}(x) = 0.$$

Then there is some v that solves

$$L\bar{u}(x) + Lv(y) - f(y) = \bar{F}(\bar{u}, x), \tag{2.3}$$

and plugging this particular choice of v back to the expansion and also in (2.2), we exactly see an effective equation as

$$\bar{F}(\bar{u}, x) = 0.$$

Moreover, in order for the approximation to make sense, and recover $u^\varepsilon \to \bar{u}$ uniformly, we require the decay condition that $\varepsilon^\sigma v(\cdot/\varepsilon)$ goes to zero uniformly.

Going back to the general linear case, we use the motivation of (2.3), with or without the Fredholm justification, to find a constant, $\bar{F}$, such that there is a periodic solution, v, of

$$[L\bar{u}(x)](y) + [Lv(y)](y) + f(y) = \bar{F}(\bar{u}, x) \quad \text{for} \quad y \in \mathbb{R}^n. \tag{2.4}$$

Solving this problem is referred to (in this work) as solving the "true corrector" equation, and is often called the cell problem in other works. It simply means we have identified the equation which must be satisfied in order that the function v would be a σ-order correction to the function $\bar{u}$ nearby x in order that the expansion of u^ε remains valid.

Although we do indeed prove this "true corrector" equation will have a solution (made precise in Section 5), it is unlikely that it will have a solution in a non-periodic setting (c.f. [12], [20] for discusion in the context of Second Order equations and Hamilton-Jacobi equations, respectively).

Our first generalization will be removing the restriction that there should be only **one function**, v, such that its rescaling, $\varepsilon^\sigma v(\cdot/\varepsilon)$, corrects the behavior of u^ε for all ε. It is natural to allow the correction term to depend on ε as well. So we search for some **appropriate family of functions**, v^ε, such that

$$u^\varepsilon = \bar{u} + v^\varepsilon + o(\varepsilon).$$

Second, we recall that for viscosity solutions, we will need to know the value of $\bar{F}(\phi, x_0)$ for all possible test functions ϕ and locations of evaluation, x_0. Supposing that we would like to show $\bar{u}$ is a subsolution of some effective equation, $\bar{F} \geq 0$, we inherit a constraint on the possible values of $\bar{F}$. $\bar{F}$ must take values such that whenever $\bar{u} - \phi$ has a global maximum at x_0, then $\bar{F}(\phi, x_0) \geq 0$. We now face two questions: what should be the value of $\bar{F}(\phi, x_0)$, and which equation must such a v^ε satisfy in order to maintain the correct sign for the subsolution inequality, $\bar{F} \geq 0$?

For the moment, we may assume $\bar{u}$ is a smooth subsolution, $\bar{F}(\bar{u}, x) \geq 0$ (this is not true, but the actual viscosity solution argument for the proof of (1.1) does not see the difference). We go back to the corrector equation at the ε level,

$$[L\bar{u}(x)](\frac{x}{\varepsilon}) + [Lv(\frac{x}{\varepsilon})](\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}) = \bar{F}(\bar{u}, x), \tag{2.5}$$

with the goal of restricting our attention to only a small neighborhood of x_0. Moreover, if $\bar{u}$ is smooth, then by the observed uniform continuity of $[L\bar{u}(x)](x/\varepsilon)$, we can restrict our attention to fixing x at x_0 (only for the term $[L\bar{u}(x)](x/\varepsilon)$, not for all of (2.5)) and only incur a small error in the equation. By the global ordering of $\phi \geq \bar{u}$, the fact that $\bar{u}(x_0) = \phi(x_0)$, and the ellipticity of L, we can conclude that

$$\begin{aligned}&[L\phi(x_0)](\frac{x}{\varepsilon}) + [Lv^\varepsilon(x)](\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}) \geq \\ &\geq [L\bar{u}(x_0)](\frac{x}{\varepsilon}) + [Lv(\frac{x}{\varepsilon})](\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}) \geq \bar{F}(\bar{u}, x) - \delta,\end{aligned}$$

for some small δ arising from the switch to x_0 fixed from any nearby x. Now we use the same logic for setting the equation to be a constant independent of ε, and this time the decay condition giving the local convergence of $u^\varepsilon \to \bar{u}$ is simply the uniform convergence of $v^\varepsilon \to 0$. Hence if we could set the top equation equal to a constant, $\bar{F}(\phi, x_0)$, we would recover

$$\bar{F}(\phi, x_0) = [L\phi(x_0)](\frac{x}{\varepsilon}) + [Lv^\varepsilon(x)](\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}) \geq \bar{F}(\bar{u}, x) - \delta \geq -\delta,$$

by the subsolution equation for $\bar{u}$. We see $\bar{F}(\phi, x_0) \geq -\delta$ for arbitrary δ and thus would obtain the correct subsolution inequality. Under the mild restriction that we hope the same v^ε should work for both the subsolution and supersolution inequalities, we are reasonably led to find the solution of the "corrector" equation:

For each ϕ and x_0, find a unique constant, $\bar{F}(\phi, x_0)$ such that there is a family of solutions, $\{v^\varepsilon\}$, solving

$$[L\phi(x_0)](\frac{x}{\varepsilon}) + [Lv^\varepsilon(x)](\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}) = \bar{F}(\phi, x_0) \quad \text{for } x \in B_1(x_0), \tag{2.6}$$

subject to the compatibility condition $\|v^\varepsilon\|_\infty \to 0$. (The appearance of $B_1(x_0)$ is merely for concreteness, for the equation is only actually used for x in a very small ball near x_0, based on the smoothness of $\bar{u}$ and ϕ.)

In the general case, including the fully nonlinear F in (1.3), we are led to the solution of **the corrector equation**:

for each ϕ **and** x_0 fixed, we must identify a **unique** $\bar{F}$ such that the unique solutions of

$$\begin{cases} F_{\phi,x_0}(v^\varepsilon, \frac{y}{\varepsilon}) = \bar{F}(\phi, x_0) & \text{in } B_1(x_0) \\ w^\varepsilon(y) = 0 & \text{on } B_1(x_0)^c. \end{cases} \tag{2.7}$$

also obeys the correct **decay** property in ε, namely

$$\lim_{\varepsilon\to 0} \max_{B_1(x_0)} |v^\varepsilon| = 0, \tag{2.8}$$

where we have used the notation similar (2.6) as

$$F_{\phi,x_0}(v, \frac{y}{\varepsilon}) = \inf_\alpha \sup_\beta \left\{ f^{\alpha\beta}(\frac{y}{\varepsilon}) + [L^{\alpha\beta}\phi(x_0)](\frac{y}{\varepsilon}) + [L^{\alpha\beta}v(\frac{y}{\varepsilon})][\frac{y}{\varepsilon}] \right\}. \tag{2.9}$$

The appearance of the Dirichlet problem is simply to have a unique family, w^ε, with which to work, and the choice of $B_1(x_0)$ is simply to indicate that we will work locally near x_0 instead of globally.

Making the above heuristic arguments rigorous for the case of viscosity solutions is the work of the Perturbed Test Function Method used by Evans in [17], [18], and will be found in this note in Section 4.

For completeness, we will in Section 5 prove that true correctors for this equation do exist. That is we prove the existence of a unique constant, $\bar{F}(\phi, x_0)$, such that there exists a global periodic solution of the true corrector equation:

$$F_{\phi,x_0}(w, y) = \bar{F}(\phi, x_0) \quad \text{in } \mathbb{R}^n.$$

In light of the rescaling dictated by (1.6), such a periodic solution can be rescaled to solve (2.7) and (2.8), excluding the boundary conditions. In Section 5 the true correctors will give a convenient inf-sup formulation for the value of the effective operator $\bar{F}(\phi, x_0)$. It is useful to point out that both methods have their advantages and disadvantages. Such a comparison will be made in Section 6.

The key ideas we employ for investigating (2.7) and (2.8) originate in [12]. In order to find such a particular choice of $\bar{F}$ in (2.7) that will also give the correct decay, (2.8), we relax the goal slightly to the new one of simply finding a relationship between the choice of a generic right hand side of (2.7) and the limit of the functions v^ε. We therefore consider, for l fixed, the unique solutions (for each $\varepsilon > 0$) of

$$\begin{cases} F_{\phi,x_0}(w_l^\varepsilon, \frac{y}{\varepsilon}) = l & \text{in } B_1(x_0) \\ w_l^\varepsilon(y) = 0 & \text{on } B_1(x_0)^c. \end{cases} \tag{2.10}$$

The goal is to show that by manipulating the choice of l, we can obtain the desired behavior on the functions $(w_l^\varepsilon)_{\varepsilon>0}$. We remark that for l negative enough, the function

$$P^+(x) = (1 - |x|^2)^2 \cdot \mathbb{1}_{B_1}(x)$$

is a subsolution of the equation (we have taken $x_0 = 0$ for the sake of presentation). By comparison with P^+, it follows that $\liminf(w_l^\varepsilon) > 0$ when l is negative enough. On the other hand, for l large enough the function

$$P^-(x) = -(1 - |x|^2)^2 \cdot \mathbb{1}_{B_1}(x)$$

is a supersolution of this equation (again, we have taken $x_0 = 0$). Hence $\limsup(w_l^\varepsilon) < 0$, by comparison with P^- whenever l is positive enough. We will be able to satisfy (2.8) if we can exhibit some l such that simultaneously $\limsup(w_l^\varepsilon) \leq 0$ and $\liminf(w_l^\varepsilon) \geq 0$ (there are also uniform Hölder estimates for w^ε, and hence the uniform convergence). Thus the hope is that for some careful choice of an intermediate value of l, we can exactly balance both behaviors. This desire to characterize the link between the choice of l and the limiting behavior of w_l^ε leads to using the obstacle problem for the analysis of the solutions w_l^ε. The conclusion of the analysis regarding (2.7) and (2.8) appears as Proposition 3.9 in Section 3.

It may be helpful to point out that although this is indeed the same as the approach of [12] in spirit, the execution is in fact a bit different. In [12], the sets in the space of polynomials where $\bar{F}(P) \leq 0$ and $\bar{F}(P) \geq 0$ are identified, and hence the equation is know in the sense of viscosity solutions. In our situation a slightly different approach is taken; we will instead directly assign a value to the operator $\bar{F}(\phi, x_0)$ for all appropriate test functions, ϕ, and $x_0 \in D$.

2.2. **Notation.** We conclude this section with a few remarks about notation. We use the notation from [10] for $C^{1,1}(x_0)$ to be the collection of all functions, ϕ, that satisfy for some $v \in \mathbb{R}^n$ and $m > 0$ fixed (and depending on ϕ),

$$|\phi(x) - \phi(x_0) - v \cdot (x - x_0)| \leq m\,|x - x_0|^2$$

for all x in a neighborhood of x_0. We also use the standard notation for the half-relaxed upper and lower limits of a sequence of functions, say $\{u^\varepsilon\}_{\varepsilon>0}$, respectively as

$$(u^\varepsilon)^*(x) = \lim_{\varepsilon\to 0} \sup_{\{\delta\leq\varepsilon,\ |x-y|\leq\varepsilon\}} u^\delta(y); \quad (u^\varepsilon)_*(x) = \lim_{\varepsilon\to 0} \inf_{\{\delta\leq\varepsilon,\ |x-y|\leq\varepsilon\}} u^\delta(y).$$

It is important to note that if $(u^\varepsilon)^* = (u^\varepsilon)_* = u$, then this implies $u^\varepsilon \to u$ locally uniformly.

We will frequently be using cubes and balls in $\mathbb{R}^n$. By the notation $B_r(x)$ we mean the ball of radius r, centered at x. By the notation $Q_r(x)$ we mean the cube with a length of side of $2r$, centered at x.

At many points we will have the need to use special functions that take the place of $1-|x|^2$ and $|x|^2-1$. We simply need to truncate them to account for their lack of integrability at infinity. We will use:

$$p^{+,z_0}(x) = \max(1 - |x - z_0|^2, 0) \tag{2.11}$$

and

$$p^{-,z_0}(x) = \min(-1 + |x - z_0|^2, 0). \tag{2.12}$$

When $z_0 = 0$, we simply write p^+ and p^-. Occasionally we may encounter the need for a function that is smooth throughout the domain, and the sign of M^+ or M^- remains the same on the whole domain. In such a situation it will be useful to have the functions above but with supports different from B_1, we hence define

$$p^+_R(x) = p^+(\frac{x}{R}) \quad \text{and} \quad p^-_R(x) = p^-(\frac{x}{R}).$$

Lastly, we would like to point out that all inequalities regarding sub and super solutions of any of the equations mentioned will follow the convention of [11] and [10]; that is u is a subsolution of an equation if

$$F(u, x) \geq 0.$$

3. The Solution of The "Corrector" Problem

The goal of this section is to show that there does indeed exist a unique choice of $\bar{F}(\phi, x_0)$ such that the solution of (2.7) also satisfies (2.8). As mentioned in Section 2 we investigate, for l fixed, the solutions of

$$\begin{cases} F_{\phi,x_0}(w^\varepsilon_l, \frac{y}{\varepsilon}) = l & \text{in } Q_1(x_0) \\ w^\varepsilon_l(y) = 0 & \text{on } Q_1(x_0)^c. \end{cases} \tag{3.1}$$

Again, the idea is to find a relationship between the choice of l and the possible limits for $(w^\varepsilon_l)_*$ and $(w^\varepsilon_l)^*$. The main observation will be that the solution to obstacle problem for the equation $F(u, x/\varepsilon) \leq l$ carries enough information in its contact set with the obstacle 0 to be able to extract information about the possible limits for w^ε_l. Here we switched to cube, Q_1,for ease of writing the proofs. This is not necessary for the final results, but it is much easier to use cubes for (2.7) to exploit the fact that they match up with $\mathbb{Z}^n$ nicely. Again, we recall from Section 2 that **the goal will be to find some appropriate choice of l that will lie just at the borderline of those l that give**

$(w_l^\varepsilon)_* \geq 0$ **and those that will give** $(w^\varepsilon)^* \leq 0$, and hence the correct choice should exhibit both inequalities simultaneously.

We now introduce the obstacle problem which will be used to identify the correct choice of l in (2.7) (the correct l will be used as the definition of $\bar{F}$). For any set, A, consider the function U_A^l defined as the least supersolution of the operator F_{ϕ,x_0} that is above zero in A:

$$U_A^l = \inf \{u : F_{\phi,x_0}(u, y) \leq l \text{ in } A \text{ and } u \geq 0 \text{ in } \mathbb{R}^n\}.$$

It will also be useful to have the rescaled function for the equation set in a scaled domain, εA, with oscillatory coefficients:

$$u_A^{\varepsilon,l} = \inf \{u : F_{\phi,x_0}(u, \frac{y}{\varepsilon}) \leq l \text{ in } \varepsilon A \text{ and } u \geq 0 \text{ in } \mathbb{R}^n\}.$$

We remark that in light of (1.6), the relationship between U_A^l and $u_A^{\varepsilon,l}$ is given by

$$u_A^{\varepsilon,l}(x) = \varepsilon^\sigma U_A^l(\frac{x}{\varepsilon}). \tag{3.2}$$

The main properties of U_A^l we will use are found in the Appendix as Lemmas 7.6, 7.7, 7.8, and 7.9.

It will in fact be possible to look at the limiting behavior of the functions, w_l^ε, solving (3.1) by using a dichotomy of the possible behaviors of the obstacle solutions, $U_{Q_{1/\varepsilon}}^l$, in the cubes $Q_{1/\varepsilon}$. That is,

$$U_{Q_{1/\varepsilon}}^l = \inf \{u : F_{\phi,x_0}(u, y) \leq l \text{ in } Q_{1/\varepsilon} \text{ and } u \geq 0 \text{ in } \mathbb{R}^n\}, \tag{3.3}$$

$$u_{Q_1}^{\varepsilon,l} = \inf \{u : F_{\phi,x_0}(u, \frac{y}{\varepsilon}) \leq l \text{ in } Q_1 \text{ and } u \geq 0 \text{ in } \mathbb{R}^n\}. \tag{3.4}$$

The two possibilities we consider are the following dichotomy for the functions $U_{Q_{1/\varepsilon}}^l$ defined in (3.3).

dichotomy:

(i) For all $\varepsilon > 0$, $U_{Q_{1/\varepsilon}}^l = 0$ for at least one point in **every** complete cell of $\mathbb{Z}^n$ contained in $Q_{1/\varepsilon}$.

(ii) There exists some ε_0 and some cell, C_0, of $\mathbb{Z}^n$ such that $U_{Q_{1/\varepsilon_0}}^l(y) > 0$ for all $y \in C_0$.

The benefit of the observed dichotomy is that it does indeed identify the possibly limits for w_l^ε as described in Lemmas 3.1 and 3.4

Lemma 3.1. *If (ii) of the dichotomy occurs, then* $(w_l^\varepsilon)_* \geq 0$.

Lemma 3.1 will be a direct consequence of the following lemma describing the behavior of the functions $u^{\varepsilon,l}_{Q_1} - w^{\varepsilon}_l$.

Lemma 3.2. *If (ii) of the dichotomy occurs, then* $(u^{\varepsilon,l}_{Q_1} - w^{\varepsilon}_l)^* \leq 0$.

We first assume Lemma 3.2, and prove Lemma 3.1. Lemma 3.2 will then be proved below.

Proof of Lemma 3.1. We use the properties of the obstacle and free solutions from Lemma 3.2, combined with the fact that $u^{\varepsilon,l}_{Q_1} \geq 0$ by definition. We have

$$-w^{\varepsilon}_l = (0 - w^{\varepsilon}_l) \leq (u^{\varepsilon,l}_{Q_1} - w^{\varepsilon}_l)$$

Thanks to Lemma 3.2, after taking the upper limits

$$(-w^{\varepsilon})^* = -(w^{\varepsilon}_l)_* \leq 0.$$

□

Proof of Lemma 3.2. The main idea of this proof is based on the monotonicity property of the obstacle problem, Lemma 7.9. That is, if $A \subset B$, then $U^l_A \leq U^l_B$. Thus, as soon as (ii) occurs, we know that $U^l_{Q_r} > 0$ on C_0 for all $r > 1/\varepsilon_0$. Furthermore by periodicity and Lemma 7.8, $U^l_{Q_r} > 0$ on all translates of C_0, $C_0 + z$, whenever r is large enough so that $Q_{1/\varepsilon_0} + z \subset Q_r$.

Thus we can build a set of the translates of C_0, which we will call C_N:

$$C_N = \bigcup_{\substack{|z| \leq N\sqrt{n} \\ z \in \mathbb{Z}^n}} (C_0 + z)$$

(recall n is the dimension). By construction, we have $U^l_{Q_{1/\varepsilon}} > 0$ on C_N whenever $1/\varepsilon > N + 1$. Moreover, we notice that for $r_N = 1/\varepsilon_0 + N + 1$, we have control over the ratio of the volumes:

$$\frac{|C_N|}{|Q_{r_N}|} = \frac{(1 + 2N)^n}{r_N^n} = \frac{(1 + 2N)^n}{(1/\varepsilon_0 + 2N + 1)^n} \to 1, \text{ as } N \to \infty.$$

Now we rescale back to the obstacle problem on Q_1. We know that inside Q_1, the rescaled version of C_N will be one cube that eventually fills up the entire volume of Q_1. Precisely, given any $\delta > 0$, there is $\varepsilon(\delta)$ such that for each $\varepsilon < \varepsilon(\delta)$ there exists a **connected cube**, C_ε, contained in Q_1 such that $u^{\varepsilon,l}_{Q_1} > 0$ on C_ε and

$$\frac{|C_\varepsilon|}{|Q_1|} \geq 1 - \delta,$$

whenever $\varepsilon < \varepsilon(\delta)$. Therefore if K_ε is the contact set for $u^{\varepsilon,l}_{Q_1}$, then $K_\varepsilon \subset Q_1 \setminus C_\varepsilon$.

We will exploit the fact that since $u^{\varepsilon,l}_{Q_1} > 0$ in C_ε, it follows that actually $u^{\varepsilon,l}_{Q_1}$ is a true solution to $F(u^{\varepsilon,l}_{Q_1}, x/\varepsilon) = l$ in C_ε. Thus we may compare the two functions $u^{\varepsilon,l}_{Q_1}$ and w^ε_l:

$$\sup_{C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l) \le \sup_{\mathbb{R}^n \setminus C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l).$$

In light of the fact that both functions are uniformly Hölder continuous in the closure of Q_1 by Theorem 7.3 and Lemma 7.6, and both have the same boundary data of 0 on $\mathbb{R}^n \setminus Q_1$, we can conclude that the supremum reduces to a narrow strip at the inside edge of Q_1

$$\sup_{\mathbb{R}^n \setminus C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l) \le C\left(\max_{x\in\partial C_\varepsilon}(d(x,\partial Q_1))\right)^\gamma.$$

Therefore, in the worst case scenario, owing to the Hölder continuity at the boundary of Q_1 we would have

$$\sup_{\mathbb{R}^n \setminus C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l) \le \tilde{C}(\delta^{1/n})^\gamma.$$

Finally this gives

$$\begin{aligned}\sup_{Q_1}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l) &\le \sup_{C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l) + \sup_{Q_1\setminus C_\varepsilon}(u^{\varepsilon,l}_{Q_1} - w^\varepsilon_l)\\ &\le \tilde{C}(\delta^{1/n})^\gamma.\end{aligned}$$

After taking the upper limit, we conclude the result because $\delta > 0$ was arbitrary. □

Remark 3.3. We note that Lemma 3.2 simply says that under the condition of (ii), the solution of the obstacle problem and the solution of the free problem become exactly the same in the limit. In the second order case, this will happen under much more general circumstances. In fact, the difference, $u^{\varepsilon,l}_{(Q_1)} - w^\varepsilon_l$, (in both the local and nonlocal cases) is a subsolution of an equation with the maximal operator, M^+, and right hand side given as a the characteristic function of the contact set between $u^{\varepsilon,l}_{(Q_1)}$ and the obstacle, $y = 0$. A result which is sensitive enough to estimate the supremum of subsolutions to such equations in terms of a measure theoretic quantity of the right hand side (the L^n norm in the second order case) is precisely what can be used to achieve the outcome of Lemma 3.2 in a more general scenario. In the second order case, this is exactly the Aleksandroff-Bakelman-Pucci estimate– the interested reader should consult [11], Chapter 3, and [12], Theorem 2.1 with its proof and discussion. However to date, such an estimate for nonlocal operators is unknown– at least to the author.

Lemma 3.4. *If (i) of the dichotomy occurs, then* $(w^\varepsilon_l)^* \le 0$.

Proof of Lemma 3.4. This lemma is a straightforward application of (i) and rescaling back to $u^{\varepsilon,l}_{Q_1}$. The rescaling, combined with the uniform continuity of $u^{\varepsilon,l}_{Q_1}$ from Lemma 7.6 gives the result. Indeed, we always have $w^\varepsilon_l \leq u^{\varepsilon,l}_{Q_1}$ simply by the fact that w^ε_l is a subsolution of the equation. Furthermore by (i), we know that $u^{\varepsilon,l}_{Q_1} = 0$ for at least one x in every single cell of $\varepsilon\mathbb{Z}^n$ that is contained in Q_1. Therefore due to the Hölder continuity from Lemma 7.6, we have that

$$w^\varepsilon_l \leq u^{\varepsilon,l}_{Q_1} \leq C\varepsilon^\gamma,$$

where C and γ are independent of ε. Now taking the local uniform upper limit, we conclude that $(w^\varepsilon_l)^* \leq 0$. □

Lemmas 3.1 and 3.4 indicate that we can now make the correct choice of l so as to correctly balance both potential limiting behaviors of w^ε_l. The comments in Section 2 showed that for l very negative, $(w^\varepsilon_l)_* \geq 0$, and therefore, we want to choose the largest such value of l that still gives this behavior. We can now characterize $\bar{F}(\phi, x_0)$ as

$$\bar{F}(\phi, x_0) = \sup\{l : (ii) \text{ occurs for the family } U^l_{Q_{1/\varepsilon}}\}. \tag{3.5}$$

We first show that this choice exhibits the correct decay for the solution (Lemma 3.5) and then in a separate lemma that it is a unique choice (Lemma 3.7). In both results that follow, we assume without loss of generality that $x_0 = 0$. If $x_0 \neq 0$, then the proofs can be modified simply by replacing Q_1 by $Q_1(x_0)$ and $p^\pm$ by $p^{\pm,x_0}$ (see (2.11),(2.12)).

Lemma 3.5. *If $l = \bar{F}(\phi, x_0)$ then w^ε_l solving (3.1) also satisfies (2.8).*

Remark 3.6. It is useful here to note that the function $p^-_{3\sqrt{n}}$ is smooth in Q_1 and satisfies $M^-(p^-_{3\sqrt{n}})(x) \geq c > 0$ in Q_1 for some c. This constant c can be chosen to depend only on Q_1, λ, Λ, and the dimension. Moreover, for $\alpha > 0$, $\alpha p^-_{3\sqrt{n}}$ satisfies $M^-(\alpha p^-_{3\sqrt{n}})(x) \geq \alpha c > 0$. Furthermore, an analogous statement holds for $p^+_{3\sqrt{n}}$ and M^+. These properties will be useful in proving multiple statements to follow.

Proof of Lemma 3.5. The stated limit will be proved in two pieces. First it will be shown that $(w^\varepsilon_l)^* \leq 0$ and then it will be shown that $(w^\varepsilon_l)_* \geq 0$.

We will first show that $(w^\varepsilon_l)^* \leq 0$. To do so, we let $l_k > l$ and $l_k \to l$ as $k \to \infty$. The goal will be to choose $\alpha_k \to 0$ such that $\alpha_k p^+_{3\sqrt{n}} + w^\varepsilon_{l_k}$ is a supersolution of (3.1) with the right hand side l. The upper limit of w^ε_l can then be controlled by that of $w^\varepsilon_{l_k}$, which is dictated by Lemma 3.4 because (i) of the dichotomy must hold due to the fact that $l_k > l$. For ease of notation, we let

$$w_k = \alpha_k p^+_{3\sqrt{n}} + w^\varepsilon_{l_k}.$$

In order to select the correct choice of α_k, we write out the equation for w_k and use Remark 3.6:

$$\begin{aligned} F_{\phi,x_0}(w_k, \frac{y}{\varepsilon}) &\le F_{\phi,x_0}(w^\varepsilon_{l_k}, \frac{y}{\varepsilon}) + M^+(w_k - w^\varepsilon_{l_k}) \\ &\le l_k - \alpha_k c. \end{aligned}$$

We thus choose α_k such that $l_k - \alpha_k c \le l$; we note that this allows $\alpha_k \to 0$ as $l_k \to l$. With this choice, w_k is indeed a supersolution of (3.1) for l. Hence by comparison,

$$\sup_{Q_1}(w^\varepsilon_l - w_k) \le \sup_{\mathbb{R}^n \setminus Q_1}(w^\varepsilon_l - w_k) \le 0.$$

In other words,

$$\alpha_k p^+_{3\sqrt{n}} + w^\varepsilon_{l_k} \ge w^\varepsilon_l.$$

Taking upper limits, applying Lemma 3.4, and using $p^+_{3\sqrt{n}} \le 1$ we conclude that

$$\alpha_k \ge (w^\varepsilon_l)^*.$$

Here we have used the fact that $l_k > l$ and hence (i) of the dichotomy must hold for l_k. Finally, taking $\alpha_k \to 0$, we conclude that $(w^\varepsilon_l)^* \le 0$.

To prove that $(w^\varepsilon_l)_* \ge 0$, there are two possibilities. Either (ii) of the dichotomy holds for $\bar{F}$, or there exist $l_k < l$, $l_k \to l$ with (ii) holding for l_k. In the former we are done,and so we assume the latter to conclude. We repeat the same argument from above, except we take $l_k < l$ and w_k as

$$w_k = \alpha_k p^-_{3\sqrt{n}} + w^\varepsilon_{l_k}.$$

The argument uses M^- to show that w_k is a subsolution of (3.1) with the right hand side l. Then we use Lemma 3.1 instead of Lemma 3.4 to conclude that $(w^\varepsilon_l)_* \ge 0$. □

Lemma 3.7. *If l is any number such that w^ε_l solving (3.1) satisfies (2.8), then $l = \bar{F}(\phi, x_0)$.*

proof of Lemma 3.7. This lemma will be proved by showing that the possibilities of $l > \bar{F}(\phi, x_0)$ and $l < \bar{F}(\phi, x_0)$ both lead to a contradiction. We will simply denote $\bar{F}(\phi, x_0)$ as $\bar{F}$.

Suppose by contradiction that $l > \bar{F}$. We already know by Lemma 3.5 that $\lim_{\varepsilon\to 0} w^\varepsilon_{\bar{F}} = 0$. Thus the goal will be to prove that the new function w, given by

$$w = \alpha p^+_{3\sqrt{n}} + w^\varepsilon_l$$

is a subsolution of (3.1) for $\bar{F}$, and then obtain a contradiction upon taking the limit as $\varepsilon \to 0$. We note that $p^+_{3\sqrt{n}}$ satisfies $M^-(p^+_{3\sqrt{n}}) \ge -c$ in Q_1, where $c > 0$ depends only on

Q_1, λ, Λ, σ, and the dimension. Moreover, $\alpha p^+_{3\sqrt{n}}$ satisfies $M^-(\alpha p^+_{3\sqrt{n}}) \geq -\alpha c$ for $\alpha > 0$. We thus have

$$\begin{aligned} F_{\phi,x_0}(w, \frac{y}{\varepsilon}) &\geq F_{\phi,x_0}(w^\varepsilon_l, \frac{y}{\varepsilon}) + M^-(\alpha p^+_{3\sqrt{n}}) \\ &\geq l - \alpha c. \end{aligned}$$

It is now possible to choose $\alpha > 0$ small enough so that $l - \alpha c \geq \bar{F}$. Hence w is a subsolution of (3.1) with a right hand side of $\bar{F}$, and by comparison we have

$$\sup_{Q_1}(w - w^\varepsilon_l) \leq \sup_{\mathbb{R}^n \setminus Q_1}(w - w^\varepsilon_l) = \sup_{\mathbb{R}^n \setminus Q_1}(\alpha p^+_{3\sqrt{n}}) = \alpha(1 - \frac{1}{9n}). \tag{3.6}$$

Specifically, this will imply

$$\alpha p^+(0) + w^\varepsilon_l(0) \leq w^\varepsilon_{\bar{F}}(0) + \alpha(1 - \frac{1}{9n}). \tag{3.7}$$

By taking the limit as $\varepsilon \to 0$, and using the assumption on w^ε_l and Lemma 3.5 for $w^\varepsilon_{\bar{F}}$, we arrive at a contradition since

$$\alpha p^+_{3\sqrt{n}}(0) = \alpha > \alpha(1 - \frac{1}{9n}).$$

Now suppose that $l < \bar{F}$. We repeat the argument above using the fact that $M^+(p^-_{3\sqrt{n}}) \leq c$, in Q_1. Moreover, αp^- satisfies $M^+(\alpha p^-_{3\sqrt{n}}) \leq \alpha c$ for $\alpha > 0$. Because $l < \bar{F}$, it is possible to choose α such that

$$w = \alpha p^-_{3\sqrt{n}} + w^\varepsilon_l$$

is a supersolution of (3.1) with the right hand side $\bar{F}$. A similar contradiction is obtained.

□

Remark 3.8. A very useful fact that follows from Lemma 3.7 is that whenever $l < \bar{F}$, (ii) of the dichotomy must hold. This follows by contradiction. If there is some $l < \bar{F}$ such that (i) holds, there will be $\tilde{l} > l$ such that (ii) holds. However, this means that $w^\varepsilon_{\tilde{l}}$ is a subsolution of the equation that w^ε_l solves. Hence $(w^\varepsilon_l)_* \geq (w^\varepsilon_{\tilde{l}})_* \geq 0$. This, combined with the fact that $(w^\varepsilon_l)^* \leq 0$ by the contradiction assumption of (i) holding, contradicts Lemma 3.7.

We conclude this section with a synopsis of what has been done. This is summarized in the following proposition.

Proposition 3.9. *For each smooth ϕ and fixed point x_0, there exists a unique number, $\bar{F}(\phi, x_0)$, such that the solution to (2.7) also satisfies (2.8).*

4. The Effective Equation and Convergence of u^ε

This section is dedicated to proving the existence of an effective nonlocal equation and the convergence of the functions, u^ε to the function solving the limiting equation, $\bar{u}$. The convergence aspect will be a straightforward application of the perturbed test function method used for homogenization in [18].

In the previous section, it was shown that the value of $\bar{F}(\phi, x_0)$ is well defined for all smooth and appropriately integrable ϕ and any point, x_0. It remains to show that $\bar{F}$ in fact describes a nonlocal equation. We work with the definitions given in ([5], Section 2) and ([9], Section 2). To do this, we must show that $\bar{F}$ is "elliptic" with respect to the same maximal and minimal nonlocal operators as in (7.3), and we must also show that $\bar{F}(\phi, \cdot)$ is a continuous function whenever ϕ is smooth and has the correct integrability at infinity. These properties will be proved in the following two separate lemmas.

Lemma 4.1 (Ellipticity of $\bar{F}$)**.** *Suppose that u and v are $C^{1,1}(x)$ for some x and bounded on $\mathbb{R}^n$. Then*

$$M^-(u-v)(x) \le \bar{F}(u,x) - \bar{F}(v,x) \le M^+(u-v)(x),$$

where M^+ and M^- are the same extremal operators as for F, given in (7.4) and (7.5).

Proof of Lemma 4.1. First we suppose that u and v are actually $C^{1,1}$ on an entire neighborhood of x, and at the end of the proof we will remove this restriction. The statements to be proved will be nearly identical for M^+ and M^-. We will only show the inequality as it pertains to M^+, namely

$$\bar{F}(u,x) - \bar{F}(v,x) \le M^+(u-v)(x).$$

Finally, due to technicalities which arise from the nonlocal nature of the equation, we must break the proof into two further cases of u and v. Starting with the scenario in which both u and v are $C^{1,1}$ in a neighborhood of x, we make two cases. First, we have **case 1** in which both u **and** v **are identically zero outside some ball,** B_R**, and that** $(u-v)$ **has zero linear part at** x (i.e., $(u-v)(x) = 0$, and $D(u-v)(x) = 0$). Secondly, we have **case 2 in which we assume that** u **and** v **are simply** $C^{1,1}$ **in an entire neighborhood of** x.

Let us begin case 1. We proceed by contradiction, and assume that for some choice of u, v, x the inequality fails. That is, there is some $\gamma > 0$ such that

$$\bar{F}(u,x) - \bar{F}(v,x) = \gamma + M^+(u-v)(x) > M^+(u-v)(x). \tag{4.1}$$

The scaling of M^+ and the fact that $u-v$ is $C^{1,1}$ in a neighborhood of x tell us two things:

(a) how the operator acts on the shifted and rescaled function,

$$M^+(\varepsilon^{-\sigma}[u-v](x+\varepsilon(\cdot)))(y) = M^+(u-v)(x+\varepsilon y), \tag{4.2}$$

(b) and the continuity of $M^+(u-v)$,

$$M^+(u-v)(x+\varepsilon y) \to M^+(u-v)(x), \quad \text{as} \;\; \varepsilon \to 0. \tag{4.3}$$

We will use Lemma 3.5 plus the fact that $M^+(\alpha p^-_{3\sqrt{n}}) \le \alpha c$ for some $c > 0$ whenever $\alpha > 0$. To this end, we take the functions w_1^ε and w_2^ε to solve (3.1) with the right hand side respectively given by $\bar{F}(u,x)$ and $\bar{F}(v,x)$. We will show that the new function

$$w = w_2^\varepsilon + \varepsilon^{-\sigma}(u-v)(x+\varepsilon(\cdot)) + \alpha p^-_{3\sqrt{n}} \tag{4.4}$$

is a supersolution to the equation governing w_1^ε. Checking this we have for ε and α small enough

$$\begin{aligned} F_{v,x}(w,\frac{y}{\varepsilon}) &\overset{(7.3)}{\le} F_{v,x}(w_2^\varepsilon,\frac{y}{\varepsilon}) + M^+(\varepsilon^{-\sigma}(u-v)(x+\varepsilon\cdot)+\alpha p^-_{3\sqrt{n}})(y) \\ &\overset{\text{convexity}}{\le} F_{v,x}(w_2^\varepsilon,\frac{y}{\varepsilon}) + M^+(\varepsilon^{-\sigma}(u-v)(x+\varepsilon\cdot))(y) + M^+(\alpha p^-_{3\sqrt{n}})(y) \\ &\overset{(4.2)}{\le} \bar{F}(v,x) + M^+(u-v)(x+\varepsilon y) + M^+(\alpha p^-_{3\sqrt{n}})(y) \\ &\overset{(4.3)}{\le} \bar{F}(v,x) + M^+(u-v)(x) + \frac{\gamma}{2} + \alpha c \\ &\overset{\text{choice of } \alpha c}{\le} \bar{F}(v,x) + M^+(u-v)(x) + \gamma \\ &\overset{(4.1)}{=} \bar{F}(u,x), \end{aligned} \tag{4.5}$$

where α is chosen small enough as to give $\alpha c < \gamma/2$.

Applying comparison for this equation, we must be careful with the boundary values, which do not exactly match up in this case. However, due to the assumption that $u-v$ is bounded, has compact support, is $C^{1,1}$ in an entire neighborhood of x, and has zero linear part at x, we can place a quadratic function of εy above and below $u-v$ at x. That is, we can ensure that for some large $C > 0$, for $y \in B_R$,

$$|(u-v)(x+\varepsilon y)| \le C\,|\varepsilon y|^2. \tag{4.6}$$

Now the comparison reads

$$\sup_{Q_1}(w_1^\varepsilon - w) \le \sup_{\mathbb{R}^n\setminus Q_1}(w_1^\varepsilon - w),$$

which thanks to the boundary values of w_1^ε, w_2^ε, and $\alpha p^-_{3\sqrt{n}}$, and the compact support of $(u-v)$, gives

$$\sup_{Q_1}(w_1^\varepsilon - w) \le \sup_{B_R\setminus Q_1} (\varepsilon^{-\sigma} C\,|\varepsilon z|^2 - \alpha p^-_{3\sqrt{n}}) \le \varepsilon^{2-\sigma}\tilde{C} + \alpha(1-\frac{1}{9n}).$$

Rewriting this, we see that for all $y \in Q_1$

$$w_1^\varepsilon(y) \le w_2^\varepsilon(y) + \varepsilon^{-\sigma}(u-v)(x+\varepsilon y) + \alpha p^-_{3\sqrt{n}}(y) + \varepsilon^{2-\sigma}\tilde{C} + \alpha(1-\frac{1}{9n}).$$

Now because of Lemma 3.5 and (4.6), we see that after letting $\varepsilon \to 0$ and evaluating at $y = 0$,

$$0 \le \alpha p^-_{3\sqrt{n}}(0) + \alpha(1-\frac{1}{9n}) < 0,$$

which is a contradiction. We have concluded case 1.

We now address case 2. The strategy will be to exploit the formal invariance of M^+ with respect to an affine addition to u and or v, which follows from the use of the second difference in the evaluation of $L^{\alpha\beta}$ and hence M^+. However, this is only formal, so a modification is required. We exploit the fact that the values of u and v outside of $B_R(x)$ for some large R have a very small effect on the the value of $M^+(u-v)(x)$. Specifically, M^+ can be written as

$$M^+(u-v)(x) = M^+(\tilde{u}-\tilde{v})(x) + \rho(R),$$

where $\tilde{u}$ and $\tilde{v}$ are truncated versions of u and v:

$$\tilde{u}(y) = \bar{u}(y)\chi_{B_R(x)}(y) \quad \text{and} \quad \tilde{v}(y) = v(y)\chi_{B_R(x)}(y),$$

and $\rho(R) \to 0$ as $R \to \infty$. R is chosen, depending on u and v, so that $\rho(R) \le \gamma/4$. After reducing M^+ to an integration on B_R plus an error, we must correct for the possibly nonzero linear part of $\tilde{u}-\tilde{v}$ at x. To fix this, we exploit the fact that M^+ is unchanged under the addition of an affine function restricted to B_R to the function $\tilde{u}-\tilde{v}$. Therefore, we define $U-V$ to be $\tilde{u}-\tilde{v}-l$, where

$$l(y) = (u(x)-v(x)) + (y-x)\cdot(Du(x)) - Dv(x).$$

We remark that it still holds that

$$M^+(u-v)(x) = M^+(U-V)(x) + \rho(R),$$

and hence with the choice of R

$$M^+(U-V)(x) \le M^+(u-v)(x) + \frac{\gamma}{4}.$$

Case 2 is completed as in Case 1, starting at (4.5), where the only change is that we work with the function w given by

$$w = w_2^\varepsilon + \varepsilon^{-\sigma}(U - V)(x + \varepsilon(\cdot)) + \alpha p^-_{3\sqrt{n}},$$

instead of (4.4).

Finally, we remove the original restriction on u and v being $C^{1,1,}$ on a neighborhood of x, instead of simply $C^{1,1}(x)$. Assume that u and v are only $C^{1,1}(x)$. We define the function ψ as

$$\psi(y) = \begin{cases} (u - v)(x) + v \cdot (y - x) - C\,|y - x|^2 & \text{if } |y - x| \le 1 \\ (u - v)(y) & \text{otherwise,} \end{cases}$$

where C and v can be chosen, thanks to the $C^{1,1}(x)$ and bounded nature of $u - v$, so that $\psi \le (u - v)$ on $\mathbb{R}^n$. We see that indeed ψ is $C^{1,1}$. Due to the fact that $\psi - (u - v)$ has a global maximum at x, $M^+(\psi - (u - v))(x) \le 0$. Therefore, we repeat the argument as though $u - v$ is $C^{1,1}$ on a neighborhood of x, but as a minor modification, we instead use the function

$$w = w_2^\varepsilon + \varepsilon^{-\sigma}\psi(x + \varepsilon(\cdot)) + \alpha p^-_{3\sqrt{n}},$$

in place of (4.4). Then after taking $\varepsilon \to 0$, we use the fact that

$$\begin{aligned} M^+(\psi)(x) &= M^+(u - v)(x) + M^+(\psi)(x) - M^+(u - v)(x) \\ &\le M^+(u - v)(x) + M^+(\psi - (u - v))(x) \\ &\le M^+(u - v)(x). \end{aligned}$$

Following through the remaining arguments of case 1, this now finishes the proof that $\bar{F}(u, x) - \bar{F}(v, x) \le M^+(u - v)(x)$. The proof that $\bar{F}(u, x) - \bar{F}(v, x) \ge M^-(u - v)(x)$ is done in a similar fashion. □

At this point, we are only half way done in showing that $\bar{F}$ is indeed an elliptic nonlocal operator. It remains to show that $\bar{F}$ produces a continuous function when acting on an appropriately smooth test function:

Lemma 4.2. *For $\phi \in C^{1,1}$, $\bar{F}(\phi, \cdot)$ is a continuous function.*

Proof of Lemma 4.2. Proceeding by contradiction, we assume there exists some ϕ and points x_k and x_0 with $x_k \to x_0$ but the continuity fails; namely

$$\left|\bar{F}(\phi, x_k) - \bar{F}(\phi, x_0)\right| > \delta$$

for some $\delta > 0$. After extracting a subsequence if necessary, we assume without loss of generality that

$$\bar{F}(\phi, x_k) \geq \bar{F}(\phi, x_0) + \delta. \tag{4.7}$$

As above, let w_k^ε and w_0^ε be solutions of (3.1) with right hand sides given by $\bar{F}(\phi, x_k)$ and $\bar{F}(\phi, x_0)$ respectively. The goal will be to show that the function

$$w = w_k^\varepsilon + \alpha p^+_{3\sqrt{n}}$$

is a subsolution of (3.1) with the operator F_{ϕ,x_0} and the right hand side $\bar{F}(\phi, x_0)$. We have by Lemma 7.5 (note we are arguing incorrectly as though w is a classical solution, which is easily translated to an argument for viscosity solutions)

$$\begin{aligned} F_{\phi,x_0}(w, \frac{y}{\varepsilon}) &\overset{\text{(Lemma 7.5)}}{\geq} F_{\phi,x_k}(w, \frac{y}{\varepsilon}) - \rho_\phi(|x_k - x_0|) \\ &\overset{(7.3)}{\geq} F_{\phi,x_k}(w_k^\varepsilon, \frac{y}{\varepsilon}) + M^-(\alpha p^+_{3\sqrt{n}}) - \rho_\phi(|x_k - x_0|) \\ &\overset{(2.7)}{=} \bar{F}(\phi, x_k) + M^-(\alpha p^+_{3\sqrt{n}}) - \rho_\phi(|x_k - x_0|) \\ &\overset{(4.7)}{\geq} \bar{F}(\phi, x_0) + \delta + M^-(\alpha p^+_{3\sqrt{n}}) - \rho_\phi(|x_k - x_0|). \end{aligned}$$

For n large enough and α small enough, we have

$$\delta + M^-(\alpha p^+_{3\sqrt{n}}) - \rho_\phi(|x_k - x_0|) \geq 0,$$

and hence w is a subsolution of (2.7). Now by comparison (using the same computations as in Lemma 3.7, equations (3.6) and (3.7)), we see that

$$w_k^\varepsilon(0) + \alpha p^+_{3\sqrt{n}}(0) \leq w_0^\varepsilon(0) + \alpha(1 - \frac{1}{9n}).$$

Applying Lemma 3.5, we arrive at an assertion that states $\alpha p^+_{3\sqrt{n}} \leq \alpha(1 - (1/9n))$, which is a contradiction. □

Finally it will be important to collect one more fact about the effective operator, $\bar{F}$, which was indeed the original goal of homogenization. We must show that $\bar{F}$ is translation invariant.

Lemma 4.3. *$\bar{F}$ is translation invariant in the sense that $\bar{F}(\phi, x + z) = \bar{F}(\phi(\cdot + x), z)$ for all $\phi \in C^{1,1}$.*

Proof. Looking back to (2.9) and (2.1), we see that the "frozen" linear operators are translation invariant (with respect to their freezing point of the centered difference) in the sense that

$$[L^{\alpha\beta}\phi(x + z)](y) = [L^{\alpha\beta}\phi(\cdot + z)(x)](y).$$

Now, suppose that w^ε is the solution of (2.7) with the operator $F_{\phi,x+z}$ and $l = \bar{F}(\phi, x+z)$. By Proposition 3.9, this tells us that w^ε also satisfies (2.8). By the previous line, we see that (as a result of the particular inf-sup form, (1.3))

$$F_{\phi,x+z}(w^\varepsilon, \frac{y}{\varepsilon}) = F_{\phi(\cdot+z),x}(w^\varepsilon, \frac{y}{\varepsilon}).$$

Thus w^ε also is a solution of (2.7) with the operator $F_{\phi(\cdot+z),x}$ and the right hand side $l = \bar{F}(\phi, x + z)$ and also satisfies (2.8). By the uniqueness of such an l, stated in Proposition 3.9, we conclude that $\bar{F}(\phi, x+z) = \bar{F}(\phi(\cdot + z), x)$. □

It has now been proved that the corrector equation can indeed be used to define an "elliptic", nonlocal effective equation in the sense of ([10] Section 2). Let $\bar{F}(\phi, x)$ be the nonlocal operator defined for a smooth ϕ by (3.5), let $\bar{u}$ be the solution of (1.2), and let u^ε be the solution of (1.1). We move on to proving the second part of Theorem 1.1, namely that $u^\varepsilon \to \bar{u}$ locally uniformly as $\varepsilon \to 0$. Before we can do so, we must know that the effective equation enjoys the comparison principle and hence has uniqueness of solutions. We state this in the next proposition. The proof is a straightforward application of the translation invariance of $\bar{F}$ and the ellipticity via the extremal operators from Lemma 4.1.

Proposition 4.4. *Let u be upper semicontinuous and v be lower semicontinuous, and both bounded. If $\bar{F}(u, x) \geq 0$ in D, $\bar{F}(v, x) \leq 0$ in D, and $u \leq v$ on D^c, then $u \leq v$ in $\mathbb{R}^n$.*

proof of Proposition 4.4. We only provide a brief comment on the proof because all details except the first observations are exactly contained in [10], Section 5, Theorem 5.2. We define u^α and v^α as the standard sup-convolution and inf-convolution respectively:

$$u^\alpha(x) = \sup_y \{u(y) - \frac{1}{2\alpha} |x-y|^2\}$$

and

$$v^\alpha(x) = \inf_y \{v(y) + \frac{1}{2\alpha} |x-y|^2\}.$$

The translation invariance of $\bar{F}$ implies that $\bar{F}(u^\alpha, x) \geq 0$ and $\bar{F}(v^\alpha, x) \leq 0$. The remainder of the proof follows exactly the remaining steps in [10], Theorem 5.2. □

We can now finish the proof of the main theorem. We remark that once the corrector equation, (2.7) and (2.8), has been resolved, this is a direct application of the Perturbed Test Function Method used in [18] (Theorem 3.3), but we simply include the details here for completeness.

Proof of Theorem 1.1. We only prove that $(u^\varepsilon)^*$ is a subsolution of (1.2). The proof that $(u^\varepsilon)_*$ is a supersolution follows similarly. For notational purposes, we denote $(u^\varepsilon)^*$ by u. In what follows, we use one of the equivalent definitions of solutions of (1.2) as given in [5] (Definition 1, Section 1). Specifically, we work with test functions which are globally above or below the sub or super solution, which is equivalent to that of [10] (Definition 2.2, Section 2).

Proceeding by contradiction, suppose that ϕ is smooth and $u-\phi$ attains a strict global max at x_0. We must show that

$$\bar{F}(\phi, x_0) \geq 0,$$

and so we assume it fails, namely

$$\bar{F}(\phi, x_0) \leq -\delta < 0,$$

for some $\delta > 0$. The goal will be to use Proposition 3.9 to construct a supersolution of (1.1) on a small neighborhood of x_0 and use comparison for (1.1) to contradict the strict maximum of $u - \phi$ at x_0.

Let w^ε be the solution of (2.7) in $Q_1(x_0)$ for $\bar{F}(\phi, x_0)$. We will now show that v^ε given by

$$v^\varepsilon(y) = \phi(y) + w^\varepsilon(y)$$

is in fact a supersolution of (1.1) on an appropriately restricted ball, $B_R(x_0)$, for R small enough. We argue as though w^ε were a classical ($C^{1,1}$) solution, which may not be the case. The full details appear in the Appendix,Lemma 7.10. Indeed by Lemma 7.5, we have for y restricted to $B_R(x_0)$

$$\left| F(\phi + w^\varepsilon, \frac{y}{\varepsilon}) - F_{\phi,x_0}(w^\varepsilon, \frac{y}{\varepsilon}) \right| = \left| F_{\phi,y}(w^\varepsilon, \frac{y}{\varepsilon}) - F_{\phi,x_0}(w^\varepsilon, \frac{y}{\varepsilon}) \right| \leq \rho_\phi(R),$$

and this holds anytime w^ε is $C^{1,1}$, but is **independent of the function** w^ε and y ($F_{\phi,x_0}(w^\varepsilon, y/\varepsilon)$ is defined in (2.9)). Thus restricting R small enough so that $\rho_\phi(R) - \delta/2 \leq 0$, we conclude that

$$F(\phi + w^\varepsilon, \frac{y}{\varepsilon}) \leq 0 \quad \text{in} \quad B_R(x_0).$$

Applying the comparison theorem, we see that for each ε,

$$\sup_{B_R(x_0)} (u^\varepsilon - \phi - w^\varepsilon) \leq \sup_{\mathbb{R}^n \setminus B_R(x_0)} (u^\varepsilon - \phi - w^\varepsilon).$$

Taking upper limits as $\varepsilon \to 0$ and using Lemma 3.5, we obtain

$$\sup_{B_R(x_0)} ((u^\varepsilon)^* - \phi) \leq \sup_{\mathbb{R}^n \setminus B_R(x_0)} ((u^\varepsilon)^* - \phi).$$

This contradicts the fact that the maximum of $u - \phi$ at x_0 was strict, and so we must have $\bar{F}(\phi, x_0) \geq 0$.

The proof that $(u^\varepsilon)_*$ is a supersolution of (1.2) follows analogously. It is worth pointing out that due to the uniform continuity estimates on u^ε that are independent of ε, given in Theorem 7.3, both $(u^\varepsilon)^*$ and $(u^\varepsilon)_*$ are equal to g on D^c. Thus since $(u^\varepsilon)^*$, $(u^\varepsilon)_*$, and $\bar{u}$ attain the same boundary data, (1.2) has the comparison given in Proposition 4.4, and using that $\bar{u}$ is a solution, we conclude that

$$\bar{u} \leq (u^\varepsilon)_* \leq (u^\varepsilon)^* \leq \bar{u}.$$

This implies local uniform convergence to $\bar{u}$. □

5. Solution To The True Corrector Equation

In this section, we sketch the details for proving that indeed the true corrector equation has a solution. As a consequence, we obtain an inf-sup formula for $\bar{F}$. The True Corrector existence is summarized in the proposition:

Proposition 5.1. *Let x_0 and $\phi \in C^{1,1}(x_0)$ be given. Then there is a unique number, $\hat{F}(\phi, x_0)$, such that the equation*

$$F_{\phi,x_0}(W, y) = \hat{F}(\phi, x_0) \quad \text{in } \mathbb{R}^n \tag{5.1}$$

admits a global, periodic solution, W. Moreover, $\hat{F} = \bar{F}$, where $\bar{F}$ is the unique constant from Proposition 3.9.

The main tool in proving Proposition 5.1 is a Liouville type theorem for global solutions of $F_{\phi,x_0}(u, y) = 0$ in $\mathbb{R}^n$, which we list as Propostion 7.4 in the Appendix for completeness.

Proof of Proposition 5.1. We just sketch the details because the proof will be almost exactly as that found in ([2]- Theorem II.2) and ([18]– Lemmas 2.1, 3.1).

The proof begins with an approximation to obtain (5.1). Let v_λ be the unique global solution of

$$\lambda v_\lambda + F_{\phi,x_0}(v_\lambda, y) = 0 \ \text{ in } \mathbb{R}^n.$$

(For much discussion regarding the relevance of the above approximation, the interested reader should consult [12], [16], [20], and [21].) The first observation is that the periodicity of F_{ϕ,x_0} directly translates to the periodicity of v_λ. Comparison with constant sub and super solutions tells us at least that λv_λ will be bounded. However, it will not necessarily be true that v_λ will be bounded uniformly in λ, and in fact it is expected that v_λ will grow as $\lambda \to 0$.

The standard arguments recover $\hat{F}(\phi, x_0)$ as the limit of $\lambda v_\lambda(0)$. To do so, we must first prove that the new function

$$w_\lambda = v_\lambda - v_\lambda(0),$$

which solves

$$\lambda w_\lambda + F_{\phi,x_0}(w_\lambda, y) = -\lambda v_\lambda(0) \text{ in } \mathbb{R}^n, \tag{5.2}$$

is bounded uniformly in λ. If such bounds do exist, then the regularity results– Theorem 7.2– allow to extract subsequences of w_λ, and the stability of (5.2) allows passage to the limit to obtain

$$F_{\phi,x_0}(W, y) = -\lim_{\lambda\to 0}(\lambda v_\lambda(0)). \text{ in } \mathbb{R}^n$$

Then one can show that any such constant appearing on the right hand side of (5.1) must be unique.

We now focus on the claim that w_λ are uniformly bounded. To see this, we suppose not, and assume instead that on some subsequence (still denoted by λ) $\|w_\lambda\|_\infty \to \infty$. The rescaled function,

$$W_\lambda = \frac{w_\lambda}{\|w_\lambda\|_\infty}$$

is periodic, bounded, and solves

$$\alpha W_\lambda + \inf_\alpha \sup_\beta \Big\{ \frac{f^{\alpha\beta}(y)}{\|w_\lambda\|_\infty} + \frac{1}{\|w_\lambda\|_\infty}[L^{\alpha\beta}\phi(x_0)](y) + L^{\alpha\beta}W_\lambda(y) \Big\} = \frac{\lambda v_\lambda(0)}{\|w_\lambda\|_\infty} \text{ in } \mathbb{R}^n. \tag{5.3}$$

Thus by Theorem 7.2, we know that W_λ are locally uniformly Hölder continuous and we may extract a convergence subsequence (again, denoted still by λ). Let $\bar{W}$ be a possible limit of W_λ. Stability of (5.3) and the boundedness of the terms $f^{\alpha\beta}$, $[L^{\alpha\beta}\phi(x_0)](\cdot)$, and $\lambda v_\lambda(0)$ dictate that $\bar{W}$ is a periodic global solution to

$$\inf_\alpha \sup_\beta \big\{ L^{\alpha\beta} W_\lambda(y) \big\} = 0 \text{ in } \mathbb{R}^n.$$

Again from the Liouville Property, Proposition 7.4, we know $\bar{W}$ must be a constant. This is a contradiction, however because $\bar{W}(0) = 0$, but $\|\bar{W}\|_\infty = 1$.

We conclude that w_λ is bounded, and just as with W_λ, we can extract a convergent subsequence. Let $w_\lambda \to \bar{w}$. It follows by stability of (5.2) that $\bar{w}$ solves

$$F_{\phi,x_0}(\bar{w}, y) = -\lim_{\lambda\to 0} \lambda v_\lambda(0).$$

The constant, $-\lim_{\lambda\to 0} \lambda v_\lambda(0)$, can be shown to be unique using the same steps in ([18] Lemmas 2.1 and 3.1). Hence

$$\hat{F}(\phi, x_0) = -\lim_{\lambda\to 0} \lambda v_\lambda(0).$$

To see that $\hat{F} = \bar{F}$, we notice that $v_\varepsilon = \varepsilon^\sigma W(\cdot/\varepsilon)$ is a solution to (3.1) with a right hand side given by $\hat{F}$, and we note that $\sup_{\mathbb{R}^n \setminus Q_1} |v_\varepsilon| \le C\varepsilon^\sigma$. Hence by comparison with $w^\varepsilon_{\hat{F}}$ we see that

$$\max_{Q_1} \left| w^\varepsilon_{\hat{F}} - v_\varepsilon \right| \le \max_{\mathbb{R}^n \setminus Q_1} \left| w^\varepsilon_{\hat{F}} - v_\varepsilon \right| \le C\varepsilon^\sigma,$$

and $\left| w^\varepsilon_{\hat{F}} \right| \to 0$. By the uniqueness of $\bar{F}$ from Proposition 3.9 we conclude that $\hat{F} = \bar{F}$. We conclude our sketch here. □

An immediate corollary of Proposition 5.1 is the useful inf-sup formula for $\bar{F}$.

Corollary 5.2. *Let x_0 and $\phi \in C^{1,1}(x_0)$ be given. Then the formula holds in the viscosity sense:*

$$\bar{F}(\phi, x_0) = \inf_{W \ \textit{periodic}} \sup_{y \in \mathbb{R}^n} \left(F_{\phi,x_0}(W, y) \right).$$

(That is to say that $\bar{F}(\phi, x_0)$ is the least constant, C, such that there is a periodic viscosity solution in $\mathbb{R}^n$ to $F_{\phi,x_0}(W, y) \le C$.)

The proof that $\bar{F}(\phi, x_0)$ is larger than (or equal to) the formula on the right hand side, above, is a direct consequence of Proposition 5.1 and the infimum in the formula. The fact that the formula is not strictly smaller than $\bar{F}(\phi, x_0)$ follows by contradiction using Remark 3.8, Proposition 3.9, and the scaling $\varepsilon^\sigma W(x/\varepsilon)$.

Another convenient and simple consequence of Proposition 5.1 is the preservation of linearity through the homogenization process.

Corollary 5.3. *If F is a linear operator, then so is $\bar{F}$.*

Indeed, it suffices to consider (1.1) such that in (1.3) $f^{\alpha\beta} = 0$ (there is of course no inf-sup in (1.3) in this case). Therefore by the linearity of (5.1), Corollary 5.3 follows directly from Proposition 5.1.

6. Conclusion and Open Questions For Nonlocal Equations

We conclude this note with some very brief comments, a discussion related work, and mention of open questions.

Therefore, we ask: "Does $\bar{F}$ correspond to a control problem of jump processes whose kernels are homogeneous in space?" This is more or less equivalent to asking "Can $\bar{F}$ be written as the inf-sup of linear operators associated to spatially homogeneous kernels?". In the case of second order elliptic equations, the answer to the second question is yes. This owes to the fact that ellipticity corresponds to a uniform Lipschitz condition on $\bar{F}$, and such functions can indeed be constructed from an inf-sup process involving linear functions.

The second comment is regarding a linear equation. In this context, to the best of the author's knowledge, these homogenization results are new even for the linear case. After this work was submitted, a related result by Arisawa, [3], was brought to the author's attention. The results of the present work and that of [3] only overlap in a small sense in the case that F is the fractional Laplacian, namely (1.1) reduces to

$$Lu^\varepsilon(x) = f(\frac{x}{\varepsilon}),$$

where L is

$$L\phi(x) = \int_{\mathbb{R}^n} (\phi(x+y) + \phi(x-y) - 2\phi(x))\, |y|^{-n-\sigma}\, \mathrm{dy}. \tag{6.1}$$

As pointed out in Section 2.1 of this work, the effective equation will reduce to

$$L\bar{u}(x) = \bar{f},$$

where

$$\bar{f} = \int_Q f(y)\mathrm{dy},$$

as pointed out to be a consequence of the true corrector equation and the Fredholm Alternative. Since both works resolve the true corrector equation, they are identical for this small case in which they overlap. It is worth pointing out, however, that there are still many more linear equations for (1.1) that are not given by

$$g(\frac{x}{\varepsilon})Lu^\varepsilon(x) = f(\frac{x}{\varepsilon}) \text{ or equivalently } Lu^\varepsilon(x) = f(\frac{x}{\varepsilon})/g(\frac{x}{\varepsilon}),$$

with L as (6.1). In particular kernels with anisotropic dependence on z and x will not be recognized in the form above, but they are indeed included in Theorem 1.1 here.

Further in the direction of linear equations, by Corollary 5.3, $\bar{F}$ will be a linear operator if F is. However, the methods do not immediately indicate that $\bar{F}(\phi, \cdot)$ is obtained as an integral of the second difference of ϕ against a kernel (except as in the discussion of the previous paragraph). None the less, it should still be true that the effective equation is represented by integrating the second difference against a kernel, along . Understanding the form of such an integration kernel will give description of the macroscopic behavior of the related jump processes– in the form of a functional limit theorem indicating that the ε level jump processes converge to another stochastic process whose behavior is linked with the properties of the effective nonlocal operator.

The inf-sup formula is important in many regards, and specifically for numerics, and we refer the interested reader to [19] for Hamilton-Jacobi equations. One aspect in particular is to allow for more efficient computations involving F and $\bar{F}$.

Finally, we comment on the two different methods available for proving Theorem 1.1. Although the true corrector equation, (5.1), has a solution we felt it was important to include the analysis involving the obstacle problem (Section 3). The benefits of the true corrector are twofold: the true corrector trivially gives half of the proof of the inf-sup formula in Corollary 5.2, and entirely gives Corollary 5.3.

In general, the true corrector is very particular to the periodic case. This stems from a lack of compactness when stationary ergodic F are under consideration. Moreover, at least in the case of Hamilton-Jacobi equations, it is proved that in some cases a corrector cannot exist, [20]. Therefore, it is not likely that the analysis of Section 5 will generalize to the random setting. This is important because often periodicity of F is too restrictive to be a realistic model. On the other hand, there is nothing special to the periodic setting for the obstacle method in Section 3. Therefore, it may be possible to generalize this method, indeed the method was introduced specifically for the random setting in [12]. The technical difficulty in *directly* generalizing this method lies entirely in proving Lemma 3.1 in the stationary ergodic setting. For those familiar with [12], in the the proof of Lemma 3.7, and as remarked here in Remark 3.3, it is noted that a sufficient tool would be a version of the Aleksandroff-Bakelman-Pucci estimate that is currently unavailable for the class of equations containing (1.1).

7. Appendix

In this section we collect some useful facts about solutions of equations in the same class as (1.1), including (1.2), (2.7), (3.1), and also solutions of certain obstacle problems.

7.1. **Comparison.**

The first fact we must note is the comparison theorem for sub and super solutions of (1.1).

Theorem 7.1. *Suppose that A is any open domain and $F(u,x)$ is within the class described by F1, F2a (b is not necessary), F3, and F5. If u and v are respectively bounded sub and super solutions of (1.1), for any $\varepsilon > 0$ fixed, in A, then comparison holds:*

$$\sup_{y\in A}(u(y)-v(y)) \le \sup_{y\in\mathbb{R}^n\setminus A}(u(y)-v(y)).$$

Proof of Theorem 7.1. We begin with a few remarks. There is a key feature of our equation which simplifies matters greatly, and makes the proof much simpler than general version appearing in [5]– the symmetry of $j^{\alpha\beta}(x,z)$ with respect to the z variable. This really says that the operators, $L^{\alpha\beta}$, have no gradient dependence. To clarify matters, we first work with a linear F, given as

$$\int_{\mathbb{R}^n} (\phi(x+j(x,y)) + \phi(x-j(x,y)) - 2\phi(x))\,|y|^{-n-\sigma}\,\mathrm{dy},$$

and we will generalize the the full form (1.3) at the end.

To keep the presentation simple, we first proceed as though u is a **strict subsolution**,

$$F(u,x) \geq \delta > 0,$$

and that the function $|x|^2$ is a valid test function for viscosity solutions of (1.1). In the definition we will work with, it is not a valid test function because of the lack of integrability against $|x|^{-n-\sigma}$ at infinity. These difficulties will be overcome by simply working with a truncated version of $|x|^2$ instead.

We may assume without loss of generality, by adding a constant to u or v that $u \leq v$ in $\mathbb{R}^n \setminus A$. Proceeding by contradiction we suppose that

$$\sup_{\mathbb{R}^n}\{u - v\} = S > 0.$$

Implementing the standard doubling of variables trick (c.f. [15], Section 3), we can achieve M as

$$S = \lim_{\alpha\to 0} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \{u(x) - v(y) - \frac{1}{\alpha}\,|x-y|^2\}. \tag{7.1}$$

If we let x_{α,y_α} be points which achieve the supremum in (7.1), then the key feature we use ([15], Lemma 3.1) is that

$$\frac{1}{\alpha}\,|x_\alpha - y_\alpha|^2 \to 0 \quad \text{as } \alpha \to 0. \tag{7.2}$$

Freezing variables at $y = y_\alpha$ respectively $x = x_\alpha$, (7.1) implies $u-(v(y_\alpha)+1/\alpha\,|\cdot - y_\alpha|^2)$ attains a maximum at x_α respectively $v-(u(x_\alpha)-1/\alpha\,|x_\alpha - \cdot|^2)$ attains a minimum at y_α. A very useful fact about nonlocal equations is that equation can actually be evaluated directly on u and v at x_α and y_α (see [10]– Lemma 3.3 or [5]– Proposition 2), therefore by the subsolution and supersolution properties:

$$\int\!\Big(u(x_\alpha + j(x_\alpha,z)) + u(x_\alpha - j(x_\alpha,z)) - 2u(x_\alpha)\Big)|z|^{-n-\sigma}\,\mathrm{dy} \geq \delta,$$

$$\int\!\Big(v(y_\alpha + j(y_\alpha,z)) + v(y_\alpha - j(y_\alpha,z)) - 2v(y_\alpha)\Big)|z|^{-n-\sigma}\,\mathrm{dy} \leq 0.$$

Revisiting (7.1), we notice that

$$u(x_\alpha + j(x_\alpha, z) - v(y_\alpha + j(y_\alpha, z)) - \frac{1}{\alpha} |x_\alpha - y_\alpha + j(x_\alpha, z) - j(y_\alpha, z)|^2 \leq$$
$$\leq u(x_\alpha) - v(y_\alpha) - \frac{1}{\alpha} |x_\alpha - y_\alpha|^2 ,$$

and similarly with $-jx_\alpha, z$, $-j(y_\alpha, z)$. Subtracting the two integrals and replacing the evaluation using the above inequality gives

$$\frac{1}{\alpha} \int \Big(|x_\alpha - y_\alpha + j(x_\alpha, z) - j(y_\alpha, z)|^2 + ...$$
$$+ |x_\alpha - y_\alpha - j(x_\alpha, z) + j(y_\alpha, z)|^2 - 2 |x_\alpha - y_\alpha|^2 \Big) |z|^{-n-\sigma} \, \mathrm{dy}$$
$$= \frac{1}{\alpha} \int 2 |j(x_\alpha, z) - j(y_\alpha, z)|^2 |z|^{-n-\sigma} \, \mathrm{dy} \geq \delta > 0.$$

At this point we can conclude by appealing to F5, which now says

$$0 < \delta \leq \frac{1}{\alpha} \int 2 \left| c(x_\alpha, \frac{z}{|z|}) - c(y_\alpha, \frac{z}{|z|}) \right|^2 |z|^2 |z|^{-n-\sigma} \, \mathrm{dy}$$
$$\leq \frac{1}{\alpha} C \left| c(x_\alpha, \frac{z}{|z|}) - c(y_\alpha, \frac{z}{|z|}) \right|^2 \int |z|^2 |z|^{-n-\sigma} \, \mathrm{dy}$$
$$\leq \frac{C}{\alpha} |x_\alpha - y_\alpha|^2 \int |z|^2 |z|^{-n-\sigma} \, \mathrm{dy} \to 0 \text{ as } \alpha \to 0.$$

Having taken $\alpha \to 0$, we have a contradiction, and so we know that $M \leq 0$.

We remark the conspicuous absence of the domain of integration above. This is because we used the full function, $|x - y|^2$, in the computation, which technically is only integrable against our kernel on bounded sets. The proof is concluded by using the same argument as above, but with $|x|^2$ truncated according to $\|u\|_\infty$ and $\|v\|_\infty$; the function $\Phi(x) = \min\{\|u\|_\infty + \|v\|_\infty, |x|^2\}$ will suffice, used as $\Phi((x-y)/\sqrt{\alpha})$ in the proof. Finally, to remove the strict subsolution restriction on u, we can perturb to $u + \delta p_R^{+,z_0}$ for an appropriate z_0 and R which ensure that A is strictly contained in the support of p_R^{+,z_0}. Then $u + \delta p_R^{+,z_0}$ is indeed a strict subsolution. □

7.2. **Ellipticity and Regularity.** We use the notion of equations and ellipticity following ([10], Sections 2 and 3) and ([9], Sections 1 and 2). Specifically for any two functions u and v which are $C^{1,1}(x)$, we require operators M^+ and M^- which are classically defined for u and v and

$$M^-(u - v)(x) \leq F(u, x) - F(v, x) \leq M^+(u - v)(x). \tag{7.3}$$

For equations defined via (1.3) and satisfying (1.7), M^+ and M^- can be written respectively as a supremum and an infimum of linear operators. We introduce the class of kernels and operators corresponding to linear, bounded, measurable coefficients:

$$\mathcal{K}_{\lambda,\Lambda} = \{K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \mid K \text{ satisfies (1.7)} \\ \text{and is measurable in both variables}\}$$

$$\mathcal{L}_{\lambda,\Lambda} = \{L \mid Lu(y) = \int_{\mathbb{R}^n} (u(y+z) + u(y-z) - 2u(y))K(y,z)\mathrm{dz} \text{ and } K \in \mathcal{K}_{\lambda,\Lambda}\}$$

Using (1.3), the extremal operators can be written as

$$M^+u(x) = \sup_{L\in\mathcal{L}_{\lambda,\Lambda}} \{Lu(x)\} \tag{7.4}$$

and

$$M^-u(x) = \inf_{L\in\mathcal{L}_{\lambda,\Lambda}} \{Lu(x)\}\,. \tag{7.5}$$

The regularity of solutions of (1.1) related equations play a crucial role in the homogenization. We collect the four most useful results for our work. Theorem 7.2, Proposition 7.4, and Lemma 7.5 respectively appear in or follow from ([10]– Section 12), ([10]– Section 12), and ([10]– Section 4, Lemma 4.2), and Theorem 7.3 appears as ([9] Corollary 3.4).

Theorem 7.2 (Caffarelli-Silvestre, A)**.** *If u is bounded on $\mathbb{R}^n$ and is simultaneously a subsolution and a supersolution in B_1 of respectively*

$$M^-u \le C_0 \quad \textit{and} \quad M^+u \ge -C_0 \quad \textit{in } B_1,$$

then u is uniformly γ- Hölder continuous in $B_{1/2}$ with γ depending only on the dimension, a lower bound on σ, and ellipticity:

$$[u]_{C^\gamma(B_{1/2})} \le C(\sup_{\mathbb{R}^n}\{u\} + C_0).$$

Theorem 7.3 (Caffarelli-Silvestre, B)**.** *The solutions of (1.1) and (7.8) are uniformly continuous with a modulus that only depends on λ, Λ, σ, n, the domain, the data, and $\|f^{\alpha\beta}\|_\infty$. Moreover if the data is Hölder continuous, then so is the solution with a possibly different Hölder exponent.*

Proposition 7.4. *Suppose that u is bounded in $\mathbb{R}^n$ and solves simultaneously*

$$M^-u \le 0 \quad \textit{and} \quad M^+u \ge 0 \quad \textit{in } \mathbb{R}^n.$$

Then u is a constant.

Sketch of Proof of Proposition 7.4. We can realize a Hölder bound for u in B_R by rescaling everything back to B_1. Because the right hand sides of the two extremal equations are both zero, the bound will decay. Specifically, Theorem 7.2, applied to the function $v_R(x) = u(Rx)$ in B_1 says that for any x and y in $B_{R/2}$,

$$|u(x) - u(y)| = \left|v(\frac{x}{R}) - v(\frac{y}{R})\right| \leq C \left|\frac{x-y}{R}\right|^{\gamma}.$$

Keeping x and y fixed as $R \to \infty$ gives $u(x) = u(y)$ for any x, y. □

Lemma 7.5. *The operator $F_{\phi,x}(v,y)$ is uniformly continuous in x, independent of v and y. That is there exists a modulus, ρ_ϕ, depending only on ϕ, such that if v and y are any function and any point for which $F(v,y)$ is well defined, then for any $x_1, x_2 \in \mathbb{R}^n$,*

$$|F_{\phi,x_1}(v,y) - F_{\phi,x_2}(v,y)| \leq \rho_\phi(|x_1 - x_2|),$$

independent of v and y.

Proof of Lemma 7.5. The proof of this lemma is a direct application of the results found in ([10]– Lemma 4.2), specifically it follows from the assertion that $L^{\alpha\beta}\phi(\cdot)$ is uniformly continuous, uniformly in α and β. We briefly give the outline of the main idea. We first recall the definitions:

$$F_{\phi,x_0}(v,y) = \inf_\alpha \sup_\beta \left\{ f^{\alpha\beta}(y) + [L^{\alpha\beta}\phi(x_0)](y) + L^{\alpha\beta}v(y) \right\}, \tag{7.6}$$

where the operator with "frozen" coefficients is

$$[L^{\alpha\beta}\phi(x_0)](y) = \int (\phi(x_0+z) + \phi(x_0-z) - 2\phi(x_0))K^{\alpha\beta}(y,z)\mathrm{dz},$$

and the usual linear operator is

$$L^{\alpha\beta}v(y) = \int (v(y+z) + v(y-z) - 2v(y))K^{\alpha\beta}(y,z)\mathrm{dz}.$$

Thanks to the bounds on $K^{\alpha\beta}$ from (1.7) and the result ([10]– Lemma 4.2), we know that for y fixed, $[L^{\alpha\beta}\phi(x_0)](y)$ is a uniformly equicontinuous family in α and β. Therefore, for each α, β,

$$\begin{aligned}
&\left(f^{\alpha\beta}(y) + [L^{\alpha\beta}\phi(x_1)](y) + L^{\alpha\beta}v(y)\right) - \left(f^{\alpha\beta}(y) + [L^{\alpha\beta}\phi(x_2)](y) + L^{\alpha\beta}v(y)\right)\\
&= [L^{\alpha\beta}\phi(x_1)](y) - [L^{\alpha\beta}\phi(x_2)](y)\\
&\leq \rho_\phi(|x_1 - x_2|),
\end{aligned}$$

for some modulus, ρ_ϕ. Due to the uniformity in α and β the result holds under operations of taking the infimum and supremum, and hence for F_{ϕ,x_0}. □

7.3. **The Obstacle Problem.** The solution of this problem will be referred to as U_A^l, and it is the least supersolution in the domain which is also constrained to be globally above 0:

$$U_A^l = \inf\left\{u : F(u,x) \le l \text{ in } A \text{ and } u \ge 0 \text{ in } \mathbb{R}^n\right\}. \tag{7.7}$$

We will also need to compare the obstacle solution to the solution of the same equation without the obstacle, referred to as the free equation and free solution:

$$\begin{cases} F(v,y) = l & \text{in } A \\ v = 0 & \text{on } A^c. \end{cases} \tag{7.8}$$

In the following lemmas we collect the four most important properties of U_A^l for homogenization.

Lemma 7.6. *There exists an exponent, γ, depending only on λ, Λ, σ, n, and A, such that U_A^l is Hölder continuous with exponent γ.*

Lemma 7.7. *The obstacle solution, U_A^l, is a true solution away from the contact set.*

Proof. Let B be a ball contained in the complement of the contact set of U_A^l, and let v be the unique function solving (7.8) in B with data given by U_A^l on B^c. Comparison in B says $v \le U_A^l$ in B because U_A^l is a supersolution with the same data. However, v, is in fact a global solution because of its equation in B and its equation inherited from U_A^l on B^c as data. Thus we also have $U_A^l \le v$ by the minimality of U_A^l, and so U_A^l solves (7.8) in B. □

Lemma 7.8. *The obstacle solution, U_A^l, satisfies the translation property: for any $z \in \mathbb{Z}^n$, on the set $A+z$,*

$$U_{A+z}^l = U_A^l(\cdot - z).$$

Proof. We prove this in two separate pieces. First, we will show that $U_{A+z}^l \le U_A^l(\cdot - z)$. By definition, U_A^l solves in A

$$F(U_A^l, x) \le l.$$

Therefore, by the periodicity of F (F4), we have that $U_A^l(\cdot - z)$ solves the equation in $A+z$ (where $y \in A+z$ is written as $x+z$ for $x \in A$):

$$F(U_A^l(\cdot - z), x+z) = F(U_A^l, x) \le l.$$

Hence $U_A^l(\cdot - z)$ is an admissible supersolution for the equation governing U_{A+z}^l in the set $A+z$. Therefore by the minimality of U_{A+z}^l, we see that $U_{A+z}^l \le U_A^l(\cdot - z)$ in $A+z$. The reverse inequality is proved similarly. □

Lemma 7.9. *If $A \subset B$, then $U^l_A \le U^l_B$.*

Proof of Lemma 7.6. We remark that because $A \subset B$, U^l_B satisfies the required equation in the set A, and hence is an admissible supersolution. Thus by the minimality of U^l_A, the claim follows. □

7.4. **The Perturbed Test Function Method.**

Lemma 7.10. *Let ϕ be C^2 and have the correct integrability for $\bar{F}$, and let $w^{\varepsilon}_{\bar{F}}$ be the solution to (2.7). If $\bar{F}(\phi, x_0) \le -\delta < 0$, then $\phi + w^{\varepsilon}_{\bar{F}}$ is a viscosity supersolution of $F(u, \frac{x}{\varepsilon}) = 0$ in $B_R(x_0)$ for R appropriately small, based only on ϕ.*

Proof. Let R be small enough, depending on Lemma 7.5, such that for any $y \in B_R(x_0)$ and any $v \in C^{1,1}(y/\varepsilon)$
$$\left| F_{\phi,y}(v, \frac{y}{\varepsilon}) - F_{\phi,x_0}(v, \frac{y}{\varepsilon}) \right| \le \frac{\delta}{2}.$$
Assume that $\phi + w^{\varepsilon}_{\bar{F}} - \psi$ has global minimum at y_0. Thus $w^{\varepsilon}_{\bar{F}} - (\psi - \phi)$ has a global minimum, and by the property that $w^{\varepsilon}_{\bar{F}}$ is a supersolution of (2.7), we see that
$$F_{\phi,x_0}(\psi - \phi, \frac{y_0}{\varepsilon}) \le \bar{F}(\phi, x_0).$$
Hence by the restriction of y_0 to $B_R(x_0)$, we can switch from evaluation at x_0 to y_0 and only incur a small error
$$F_{\phi,y_0}(\psi - \phi, \frac{y_0}{\varepsilon}) \le \bar{F}(\phi, x_0) + \frac{\delta}{2},$$
which exactly evaluates to
$$F(\psi, \frac{y_0}{\varepsilon}) \le \bar{F}(\phi, x_0) + \frac{\delta}{2}.$$
Now thanks to the assumption that $\bar{F}(\phi, x_0) \le -\delta$, we conclude that the new function $\phi + w^{\varepsilon}_{\bar{F}}$ satisfies
$$F(\phi + w^{\varepsilon}_{\bar{F}}, \frac{y}{\varepsilon}) \le -\frac{\delta}{2} < 0$$
in the ball $B_R(x_0)$. □

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