

INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLINEAR DIRECTIONAL DEPENDENCE

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ABSTRACT. We prove Hölder regularity results for a class of nonlinear elliptic integro-differential operators with integration kernels whose ellipticity bounds are strongly directionally dependent. These results extend those in [9] and are also uniform as the order of operators approaches 2.

1. INTRODUCTION

In this note we prove some Hölder regularity results for viscosity solutions of integro-differential equations in which the kernels defining the operators have strong directional dependence and do not need to satisfy everywhere pointwise comparison with the canonical kernel corresponding to the fractional Laplacian. One of our aims is to bring into better alignment the results which have been known for linear equations from the Probability and Potential Theory communities and those from the Nonlinear PDE community (see [Section 2](#)).

Before proceeding we mention that we have tried to collect the notation contained herein— as much as possible— in [Section 3.1](#). We also include a more detailed discussion of background in [Section 2](#). We first state our main result, and then develop the related operators and background in the remainder of [Section 1](#) and [Section 2](#).

The simplest example of the operators we study is

$$Lu(x) = \int_{\mathbb{R}^n} \delta_h^2 u(x) K(x, h) dh, \quad (1.1)$$

where $\delta_h^2 u(x) = u(x + h) + u(x - h) - 2u(x)$. The most canonical choice is

$$K(x, h) = C(n, \alpha) |h|^{-n-\alpha},$$

for some $\alpha \in (0, 2)$. For an appropriate constant $C(n, \alpha)$, this gives $L = -(-\Delta)^{\alpha/2}$, where $-(-\Delta)^{\alpha/2}$ is the operator whose Fourier multiplier is $-|\xi|^\alpha$ (see [24, Chapter I.1]). The interesting cases we target are when K is allowed to have large regions where $K(x, h)$ is not necessarily comparable to $|h|^{-n-\alpha}$ from below— instead, for each x , $K(x, \cdot)$ is only required to be in the class we call \mathcal{A}_{sec} , see [\(3.2\)](#). Furthermore we treat the case of L without assuming any regularity in the x variable. Both of these points are expanded upon in [Remarks 1.2, 1.3](#).

The nonlinear directional dependence enters the picture through the assumption that the kernels we treat need only to be above $|h|^{-n-\alpha}$ on a possibly small set as seen by

$$(2 - \alpha) \mathbb{1}_{V_\xi}(h) \frac{\lambda}{|h|^{n+\alpha}} \leq K(x, h) \leq (2 - \alpha) \frac{\Lambda}{|h|^{n+\alpha}} \quad h \in \mathbb{R}^n \setminus \{0\}. \quad (1.2)$$

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Here $V_\xi \subset \mathbb{R}^n$ is a conical set of the form $V_\xi = \{z \in \mathbb{R}^n \mid |\langle z/|z|, \xi \rangle| \geq \delta\}$ with $\xi \in \mathbb{S}^{n-1}$, and $\delta \in (0, 1)$ is fixed throughout. The direction ξ is allowed to depend on x - and hence also implicitly on u - both of which can be in a nonlinear fashion. This dependence on u can be seen in the statement and proof of Lemma 3.2 as well as the comments in Remark 1.2. The flexibility of ξ to depend on x and u presents significant challenges to proving our main result.

It is by now standard that to study regularity properties of solutions to equations involving L assuming no regularity in x , one must in some sense “treat all equations at once”, see [9, Sections 2, 3], [12, Chapter 2], [17, Chapters 9, 17], [23]. This means that rather than study directly solutions of

$$Lu(x) = f(x) \text{ in } \Omega,$$

for $f \in L^\infty(\Omega)$ and Ω a bounded domain, we instead study u which simultaneously solve the two inequalities

$$\inf_{K \in \mathcal{A}} \{L_K u(x)\} \leq C \text{ and } \sup_{K \in \mathcal{A}} \{L_K u(x)\} \geq -C \text{ in } \Omega.$$

The class of kernels \mathcal{A} is chosen (in our case, described by (3.2)), so that it will at least contain all the K under consideration (and sometimes is a much larger set if one wishes to attain further convenient properties of the extremal operators, e.g. rotational invariance). The extremal operators are given as

$$M_{\mathcal{A}}^- u(x) = \inf_{K \in \mathcal{A}} \{L_K u(x)\} \text{ respectively } M_{\mathcal{A}}^+ u(x) = \sup_{K \in \mathcal{A}} \{L_K u(x)\}.$$

By this line of argument, treating the case of L with only bounded measurable dependence in x is basically the same as treating general fully nonlinear equations

$$F(u, x) = f(x), \tag{1.3}$$

as soon as F satisfies the ellipticity assumption

$$M_{\mathcal{A}}^-(u - v)(x) \leq F(u, x) - F(v, x) \leq M_{\mathcal{A}}^+(u - v)(x). \tag{1.4}$$

(see [9, Sections 2, 3], [12]).

The program of studying regularity properties of nonlocal equations such as (1.3) was presented in [9], and here we extend those results to cover the larger class, \mathcal{A}_{sec} . Our main result is the following Hölder regularity estimate.

Theorem 1.1 (Hölder Regularity). *Let $\alpha_0 \in (0, 2)$ be fixed. Let $M_{\mathcal{A}_{sec}}^\pm$ be as defined in (3.4) and (3.5), and let $\alpha \in (\alpha_0, 2)$. There are positive constants $\beta \in (0, 1)$ and $C \geq 1$ depending only on $n, \lambda, \Lambda, \alpha_0$ and δ such that if $u \in L^\infty(\mathbb{R}^n)$ satisfies in the viscosity sense*

$$M_{\mathcal{A}_{sec}}^- u \leq C' \text{ and } M_{\mathcal{A}_{sec}}^+ u \geq -C' \text{ in } B_1(0), \tag{1.5}$$

then

$$\|u\|_{C^\beta(B_{1/2}(0))} \leq C \left(\sup_{\mathbb{R}^n} |u| + C' \right). \tag{1.6}$$

Furthermore C remains bounded as $\alpha \rightarrow 2^-$.

Remark 1.2. Allow us to explain the dependence on x and u in, e.g. (1.1), (1.2). Just as an example we will use the operator $M_{\mathcal{A}}^+$. Suppose that u and x are both fixed and that $K([u], x, \cdot)$ is chosen as an optimizer in e.g.

$$M_{\mathcal{A}}^+ u(x) = \int_{\mathbb{R}^n} \delta_h^2 u(x) K([u], x, h) dh.$$

Carrying out this procedure for each x does indeed yield a kernel say $K^*(x, h)$ which is measurable in x and for each x is in \mathcal{A} (whichever \mathcal{A} may have been chosen for $M_{\mathcal{A}}^+$). Thus the situation of (1.5) does in fact cover the special case of (1.1) where K (and hence ξ in (1.2)) has at each x an implicit (nonlinear) dependence on u and hence the direction ξ can as well.

Remark 1.3. We would like to explicitly point out that there is no harm in assuming that \mathcal{A}_{sec} contains only kernels of the form $K(h)$ and not of the form $K(x, h)$. Because the set \mathcal{A}_{sec} contains *all* such possible kernels, $K(h)$, it is certain that for each x , $K(x, \cdot)$ will be an element of \mathcal{A}_{sec} . Hence $M_{\mathcal{A}_{sec}}^{\pm}$ will indeed suffice as extremal operators for (1.1). This is in fact a common occurrence in nonlinear equations (or equations with only bounded measurable coefficients) in that the original equations may not be translation invariant, yet the extremal operators are translation invariant (and for our class, also rotationally invariant). Such invariance properties of $M_{\mathcal{A}_{sec}}^{\pm}$ are very useful to us throughout the article, and Theorem 1.1 would not hold in such generality without them.

Remark 1.4. Just as in [11, Theorem 26], Theorem 1.1 also applies to any u such that

$$u(y)(1 + |y|^{n+\alpha_0})^{-1} \in L^1(\mathbb{R}^n).$$

Remark 1.5. An important application of Theorem 1.1 is the $C^{1,\beta'}$ ($B_{1/2}$) regularity of solutions of (1.3) with $f = 0$ and a *translation invariant* F satisfying (1.4) for a special subclass $\tilde{\mathcal{A}}_{sec} \subset \mathcal{A}_{sec}$. Just as in [9] we impose the additional restriction that for all $K \in \tilde{\mathcal{A}}_{sec}$ it holds for a fixed $\rho_0 > 0$ and a uniform \tilde{C}

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(h) - K(h - z)|}{|z|} dh \leq \tilde{C} \text{ for each } |z| < \frac{\rho_0}{2}. \quad (1.7)$$

Then [9, Theorem 13.1] carries over to our setting with almost no modifications, and we conclude that also Theorem 1.1 implies $C^{1,\beta'}$ regularity for this larger class, $\tilde{\mathcal{A}}_{sec}$. The proof works because [9, Theorem 13.1] only uses two main ingredients involving the kernels: the assumption (1.7) and the ellipticity properties of $F, M_{\tilde{\mathcal{A}}_{sec}}^{\pm}$. See also [22] for more general results on $C^{1,\beta'}$ regularity.

Remark 1.6. Although we motivated Theorem 1.1 using some operators with x dependence, the uniqueness issue for viscosity solutions involving operators as (1.1), (1.3) is still open. So far uniqueness is known for translation invariant operators like (1.3)– [9, Sections 2-5]– and ones which can be written in the so-called Lévy-Ito form in [2, Section 2.2]. Typically the Lévy-Ito form can be rewritten as (1.1).

Remark 1.7. Just as in [9], additional difficulty arises from finding a proof of Theorem 1.1 in which C remains bounded as $\alpha \rightarrow 2^-$. In this article these difficulties are mostly contained in Sections 4 and 5.

Remark 1.8. In the case that $V_{\xi} = \mathbb{R}^n$ in (1.2), [9, Theorem 12.1] is contained in Theorem 1.1. We use the same methods employed therein– even some of the statements of the auxiliary results are the same between [9] and this work. However, there are significant technical difficulties which arise due to the lower bound in (1.2) holding only on a small set, V_{ξ} . These difficulties are hidden in the proofs of some of the auxiliary lemmas, and so we believe it is important to give a careful presentation of where the difficulties arise and how they are resolved. The lower bound causes minor changes to the pointwise evaluation property of subsolutions (Proposition 3.1) and the ABP substitute (Lemmas 4.1, 4.2, Theorem 4.6), and major changes to the construction of the special bump function (all of Section 5) and the validity of the Harnack inequality (see Section 7.1). The assumptions and main result of our work are also studied in [7]. Although

the class \mathcal{A}_{sec} is covered in [7], we believe our presentation of the same results is a useful contribution to the field. For example, as pointed out here in [Section 2](#) and elaborated in [Section 7.1](#), a strong version of the Harnack inequality as claimed by [7, Theorem 3.14] fails to hold when the kernels are as in (3.2).

The organization of the article is as follows. In [Section 2](#) we review some background related to [Theorem 1.1](#). In [Section 3](#) we collect notation, definitions, and preliminary results regarding (1.5). [Section 4](#) is dedicated to proving a nonlocal finite cube substitute for the Aleksandrov-Bakelman-Pucci estimate—arguably the core of most of the regularity theory for nonlinear equations. [Section 5](#) is used to construct a special bump function which is crucial to the “point-to-measure” estimates. In [Section 6](#) we prove the point-to-measure estimates and put together the remaining pieces of the proof. Finally in [Section 7](#) we present some examples, further results, and further discussion.

2. BACKGROUND

There is a rapidly growing collection of results related to [Theorem 1.1](#). We will try to focus on the type of results which only depend on the ellipticity constants, λ and Λ , as well as possibly the order, α , and we emphasize that the list of references presented is not exhaustive. As one obvious omission, we do not discuss related results for nonlocal Dirichlet forms or divergence form equations. A discussion of results in these directions can be found in [21, Section 2].

There are a few interesting distinctions to be made: whether or not $K(x, h)$ is assumed to be even in h ; whether or not the corresponding equations are linear; whether or not a Harnack inequality holds; and if the methods are probabilistic or PDE. Especially when it is not assumed that $K(x, -h) = K(x, h)$, [Theorem 1.1](#) does not apply to these equations (specified by such K). This non-symmetric class represents an important area for applications. Results obtained by probabilistic methods in many cases only capture regularity and/or Harnack’s inequality for solutions of

$$Lu(x) = 0 \text{ in } B_1. \quad (2.1)$$

In contrast, PDE techniques can usually capture the same behavior for e.g.

$$Lu(x) \text{ bounded in } B_1.$$

A more general form of (1.1) is

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+h) - u(x) - \langle \nabla u(x), h \rangle \mathbb{1}_{\{|h| \leq 1\}}) K(x, h) dh, \quad (2.2)$$

which is usually the natural (non-divergence) form of a general integro-differential operator. The reduction to (1.1) results from the extra assumption $K(x, h) = K(x, -h)$.

Regularity results (such as [Theorem 1.1](#)) as well as the Harnack inequality for linear equations with operators similar to (1.1) obtained by probabilistic methods go back at least to [6]. There (2.1) is treated assuming K is even in h and $V_\xi = \mathbb{R}^n$ in (1.2). Hölder regularity as well as a Harnack inequality are obtained. The results of [6]—both regularity and Harnack inequality—were generalized by [29], and the regularity was generalized to variable order situations in [5]. These results were generalized in [20], where regularity results were obtained for a kernel with a lower bound just as in (1.2). Finally, higher regularity in the form of Schauder type estimates were obtained in [3]. None of these results are robust as $\alpha \rightarrow 2^-$.

In the realm of PDE methods, an important result for Hölder regularity is [28], where [Theorem 1.1](#) is proved for kernels which (1.2) holds with $V_\xi = \mathbb{R}^n$, but no symmetry assumptions

are made on K , and also variable order operators are included. In [1] regularity results were obtained for kernels very similar to (1.2) by a completely different approach now called the “Ishii-Lions method” which follows [19]. The Ishii-Lions method is quite versatile, and applies more easily to different types of equations other than just uniformly elliptic ones. However, the results in [1] are not robust in α and they also depend on a modulus of continuity of K in x – as opposed to only depending on the lower and upper bounds for K – which is often not desirable for applications of Theorem 1.1 (e.g. $C^{1,\beta'}$ regularity [11, Section 13]; homogenization [26], [27]).

The first robust results appeared in [9], [10], [11], where a whole program was developed involving $C^{1,\beta'}$ and then classical regularity (the Evans-Krylov Theorem for the integro-differential setting) for fully nonlinear equations which captures all of the existing second order theory as a limit $\alpha \rightarrow 2^-$. This led to a surge in related results, and we mention a few. An important class of kernels are those for which the symmetry $K(x, -h) = K(x, h)$ is not assumed to hold; the results of [9] were extended to the non-symmetric case in [13], [14]. Via [9, Sections 10, 12], Hölder regularity for a smaller– but different class– of kernels than those treated in [9] follows as a straightforward consequence to the Aleksandrov-Bakelman-Pucci type result obtained in [18]. The subclass of (1.2) in which the direction ξ is fixed for all K was treated in [25]. This restriction actually makes the construction of a special bump function harder due to the lack of rotational invariance of the extremal operators in that instance.

An interesting feature of nonlocal equations such as (2.1) or (1.3) is that Hölder regularity and Harnack inequality no longer appear as a joined pair of results– in contrast to the local ($\alpha = 2$) theory. We record it here for later reference:

Note 2.1 (Harnack Inequality). The operators L or $M_{\mathcal{A}}^{\pm}$ (for a generic class, \mathcal{A} , not necessarily \mathcal{A}_{sec}) are said to satisfy the Harnack inequality if there exists a universal positive constant c such that for any globally non-negative u solving (2.1), respectively (1.5) in B_1 , then

$$u(x_1) \leq c(u(x_2) + C') \text{ for all } x_1, x_2 \in B_{1/2}, \quad (2.3)$$

where $C' = 0$ for the case of (2.1) and C' is given in (1.5) otherwise.

Typically in the second order case, one first proves a weak Harnack inequality ([17, Chapter 8, 9] or the L^ε Lemma [12, Lemma 4.6], [9, Theorem 10.3]) and then deduces the Harnack inequality. Then the Harnack inequality is used to prove reduction of oscillation and subsequently Hölder continuity (see [12, Chapter 4], [17, Chapter 9]). More care is needed in the nonlocal setting because it is not always true that the Harnack inequality holds. In most of the results mentioned above for the integro-differential setting, Hölder regularity is deduced directly from the point-to-measure estimates, weak Harnack / L^ε Lemma, or uniform entrance/exit time estimates for a related stochastic process.

A necessary and sufficient condition for solutions of (2.1) to satisfy the Harnack inequality is given in [8], where they also provide an example of an L such that $K \in \mathcal{A}_{sec}$ (see (3.2)) but the Harnack inequality fails. We will discuss this example in slightly more detail in Section 7. This is interesting because the Harnack inequality is proved for the fully nonlinear integro-differential case when $V_\xi = \mathbb{R}^n$ in (1.2), [9, Section 11]. The Harnack inequality is also stated to hold in [7] for a class which contains \mathcal{A}_{sec} , which cannot be true by [8, Theorem 1 and Example of p.148]. We will further discuss in Section 7 where the proof of [9, Section 11] breaks down when one considers the larger class \mathcal{A}_{sec} , (3.2).

3. PRELIMINARIES

3.1. Notation. We first collect some notations which will be used throughout this article.

$\alpha \in (\alpha_0, 2)$ is the order of the operators and $\alpha_0 \in (0, 2)$

$\delta \in (0, 1)$ is the opening of a conical sector

$$V_\xi = \left\{ z \in \mathbb{R}^n : \left| \frac{z}{|z|} \cdot \xi \right| \geq \delta \right\} \cup \{0\} \quad (3.1)$$

$$\mathcal{A}_{sec} = \left\{ K : \mathbb{R}^n \rightarrow \mathbb{R} : K(-h) = K(h), \right. \\ \left. \text{and } \exists \xi \in \mathbb{S}^{n-1} \text{ s.t. } \mathbb{1}_{V_\xi}(h) \frac{\lambda(2-\alpha)}{|h|^{n+\alpha}} \leq K(h) \leq \frac{\Lambda(2-\alpha)}{|h|^{n+\alpha}} \right\} \quad (3.2)$$

$u \in C^{1,1}(x)$, if $\exists v \in \mathbb{R}^n$ and $A > 0$ s.t.

$$|u(x+h) - u(x) - \langle v, h \rangle| \leq A|h|^2 \text{ for } h \text{ small enough}$$

$$\delta_h^2 u(x) = u(x+h) + u(x-h) - 2u(x) \quad (3.3)$$

$$L_K u(x) = \int_{\mathbb{R}^n} \delta_h^2 u(x) K(h) dh \\ M_{\mathcal{A}_{sec}}^+ u(x) = \sup_{K \in \mathcal{A}_{sec}} \{L_K u(x)\} \quad (3.4)$$

$$M_{\mathcal{A}_{sec}}^- u(x) = \inf_{K \in \mathcal{A}_{sec}} \{L_K u(x)\} \quad (3.5)$$

$$\frac{1}{2}V_\xi = \left\{ z \in \mathbb{R}^n : \left| \frac{z}{|z|} \cdot \xi \right| \geq \frac{\delta+1}{2} \right\} \quad (3.6)$$

$$Q_l(x_0) = \{x \in \mathbb{R}^n : |x - x_0|_\infty < \frac{l}{2}\}$$

$$tQ_l(x_0) = \{x \in \mathbb{R}^n : |x - x_0|_\infty < \frac{tl}{2}\}$$

$$B_l(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < l\}$$

$$\mu(dh) = |h|^{-n-\alpha} dh$$

dS_r, dS are respectively surface measure on the spheres $\partial B_r, \partial B_1$

We use $|\cdot|$ for the absolute value, the Euclidean norm, and the n -dimensional Lebesgue measure at the same time. Throughout this article $\Omega \subset \mathbb{R}^n$ denotes a bounded domain. For cubes and balls such that $x_0 = 0$ we write Q_l instead of $Q_l(0)$ and similarly for B_l . Note that the following implications hold:

$$B_{1/2} \subset Q_1 \subset Q_3 \subset B_{\frac{3\sqrt{n}}{2}} \subset B_{2\sqrt{n}}.$$

Note that $2Q_1 \neq \{2x \in \mathbb{R}^n : x \in Q_1\}$.

3.2. Definitions. We use the definitions and basic properties of viscosity solutions from [9, Sections 3, 4, 5] and [2] for (1.5). The curious reader should also see the presentation and references from the local theory in [16].

3.3. Pointwise Evaluation. A very useful feature of the viscosity solution theory for integro-differential equations is that viscosity subsolutions themselves— not only the test functions— can be used to evaluate their corresponding equation classically at all of the points where the equation is expected to hold in the weak sense— that is at all points where the subsolution can be touched from above by a smooth test function. Although this aspect of the theory is not complicated, one must proceed carefully because in the general situation of $K \in \mathcal{A}_{sec}$ one no

longer has the convenient result from previous cases (cf. [9, Lemma 4.3]) which says that at points where u is touched from above by a smooth function, the following extremely strong result holds:

$$\int_{\mathbb{R}^n} |\delta_h^2 u(x)| |h|^{-n-\alpha} dh < \infty.$$

Although this convenient regularity on u is false in the general cases treated herein (see [Section 7.2](#)), the definitions of M^+ and viscosity subsolutions together guarantee enough regularity on u so that the equation can still be evaluated classically at points of being touched by a test function. The main result we use is [Proposition 3.1](#) (cf. [9, Lemma 4.3]).

Since in this work, we only need this property for the equation

$$M_{\mathcal{A}_{sec}}^+ u(x) \geq -f(x) \text{ in } B_1 \quad (3.7)$$

we only state the results as they pertain to this particular one. Here we assume for the sake of simplicity that $f \in C(\overline{B_1})$. We also note that the pointwise evaluation holds for general F which are elliptic with respect any class which satisfies the upper bound of [\(1.2\)](#). We include further discussion on this matter in [Section 7](#).

Proposition 3.1 (Pointwise Evaluation). *Assume u solves [\(3.7\)](#) in the viscosity sense. If for $\phi \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$, $u - \phi$ has a global maximum at $x \in B_1$, then $M_{\mathcal{A}_{sec}}^+$ can be evaluated classically on u at x , and $M_{\mathcal{A}_{sec}}^+ u(x) \geq -f(x)$.*

Although the proof of [Proposition 3.1](#) is straightforward, we break it into two separate lemmas for clarity.

Lemma 3.2 (Extremal Formula). *Assume $u \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$. Then we have the formulas*

$$M_{\mathcal{A}_{sec}}^+ u(x) = \sup_{\xi \in \mathbb{S}^{n-1}} \left((2-\alpha) \int_{\mathbb{R}^n} (\Lambda(\delta_h^2 u(x))^+ - \lambda(\delta_h^2 u(x))^- \mathbb{1}_{V_\xi}(h)) \mu(dh) \right) \quad (3.8)$$

$$M_{\mathcal{A}_{sec}}^- u(x) = \inf_{\xi \in \mathbb{S}^{n-1}} \left((2-\alpha) \int_{\mathbb{R}^n} (\lambda(\delta_h^2 u(x))^+ \mathbb{1}_{V_\xi}(h) - \Lambda(\delta_h^2 u(x))^-) \mu(dh) \right) \quad (3.9)$$

Remark 3.3. [Lemma 3.2](#) is not absolutely necessary to prove [Proposition 3.1](#), but we think it is useful in its own right and hence include it here.

Lemma 3.4 (Upper Semi-continuity). *For x fixed, the functional*

$$v \mapsto M_{\mathcal{A}_{sec}}^+ v(x)$$

is upper semicontinuous with respect to pointwise convergence in h of $\delta_h^2 v(x)$ in the space of functions

$$\{v : v(h) \leq \phi(h) \text{ for all } h \text{ and } v(x) = \phi(x)\}, \quad (3.10)$$

for some fixed $\phi \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$.

Remark 3.5. We note that for a fixed u in the set [\(3.10\)](#), the map

$$\xi \mapsto \int_{\mathbb{R}^n} (\Lambda(\delta_h^2 u(x))^+ - \lambda(\delta_h^2 u(x))^- \mathbb{1}_{V_\xi}(h)) \mu(dh)$$

is continuous. Hence the sup in [\(3.8\)](#) is achieved for any such u . This is not necessary for the arguments below, but useful for reference and simplification.

Proof of Lemma 3.2. We only prove the case for $M_{\mathcal{A}_{sec}}^+$. Let $\xi(u, x) \in \mathbb{S}^{n-1}$ be the optimizer for the right hand side of (3.8), see Remark 3.5. We note that for u fixed, the kernel,

$$K_u(h) = (2 - \alpha) \left(\lambda \mathbb{1}_{\{\delta_h^2 u(x) < 0\}}(h) \mathbb{1}_{V_{\xi(u)}}(h) + \Lambda \mathbb{1}_{\{\delta_h^2 u(x) > 0\}}(h) \right) |h|^{-n-\alpha},$$

is in the set \mathcal{A}_{sec} . Hence by the definition of $M_{\mathcal{A}_{sec}}^+$, (3.4), we see that the left side of (3.8) is greater or equal to the right hand side.

Now for the reverse inequality. Let K be any other kernel in \mathcal{A}_{sec} , and let ξ be the direction for the lower bound of K in (1.2).

$$\begin{aligned} \int_{\mathbb{R}^n} \delta_h^2 u(x) K(h) dh &= \int_{\mathbb{R}^n} ((\delta_h^2 u(x))^+ - (\delta_h^2 u(x))^-) K(h) dh \\ &\leq (2 - \alpha) \int_{\mathbb{R}^n} (\Lambda (\delta_h^2 u(x))^+ - \lambda (\delta_h^2 u(x))^- \mathbb{1}_{V_{\xi}}(h)) \mu(dh) \\ &\leq \sup_{\xi \in \mathbb{S}^{n-1}} \left((2 - \alpha) \int_{\mathbb{R}^n} (\Lambda (\delta_h^2 u(x))^+ - \lambda (\delta_h^2 u(x))^- \mathbb{1}_{V_{\xi}}(h)) \mu(dh) \right) \end{aligned}$$

□

Proof of Lemma 3.4. Assume that $\delta_h^2 v_m(x) \rightarrow \delta_h^2 v(x)$ pointwise in h . Let ξ_m be optimizers of (3.8) for v_m (see Remark 3.5), and let ξ_0 be any accumulation point of $\{\xi_m\}$. We note that v_m satisfying (3.10) implies $M_{\mathcal{A}_{sec}}^+ v_m(x) \in [-\infty, C(\phi)]$, and so either an optimizing ξ exists or $M_{\mathcal{A}_{sec}}^+ v_m(x) = -\infty$, in which case we can assign ξ_m as any element of \mathbb{S}^{n-1} . Then we have the pointwise convergence for both

$$(\delta_h^2 v_m(x))^- \mathbb{1}_{V_{\xi_m}}(h) |h|^{-n-\alpha} \rightarrow (\delta_h^2 v(x))^- \mathbb{1}_{V_{\xi_0}}(h) |h|^{-n-\alpha}$$

and

$$(\delta_h^2 v_m(x))^+ |h|^{-n-\alpha} \rightarrow (\delta_h^2 v(x, h))^+ |h|^{-n-\alpha}.$$

This implies that

$$\begin{aligned} &\limsup_{m \rightarrow \infty} M_{\mathcal{A}_{sec}}^+ v_m(x) \\ &= (2 - \alpha) \limsup_{m \rightarrow \infty} \left(-\lambda \int_{\mathbb{R}^n} (\delta_h^2 v_m(x))^- \mathbb{1}_{V_{\xi_m}}(h) |h|^{-n-\alpha} dh + \Lambda \int_{\mathbb{R}^n} (\delta_h^2 v_m(x))^+ |h|^{-n-\alpha} dh \right) \\ &\leq (2 - \alpha) \left(-\lambda \int_{\mathbb{R}^n} (\delta_h^2 v(x))^- \mathbb{1}_{V_{\xi_0}}(h) |h|^{-n-\alpha} dh + \Lambda \int_{\mathbb{R}^n} (\delta_h^2 v(x))^+ |h|^{-n-\alpha} dh \right) \\ &\leq M_{\mathcal{A}_{sec}}^+ v(x), \end{aligned} \tag{3.11}$$

where we applied respectively Fatou's Lemma and dominated convergence (hence using (3.10)) to the first and second terms of (3.11). □

Proof of Proposition 3.1. We define the auxiliary functions

$$\phi_r(y) = \begin{cases} \phi(y) & \text{if } y \in B_r(0) \\ u(y) & \text{if } y \in \mathbb{R}^n \setminus B_r(0). \end{cases}$$

Proposition 3.1 now follows directly from the facts that $u - \phi_r$ has a global maximum at x , u is a viscosity subsolution of (3.7), $\delta_h^2 \phi_r(x, \cdot) \rightarrow \delta_h^2 u(x, \cdot)$ pointwise, and Lemma 3.4. In particular

$$-f(x) \leq M_{\mathcal{A}_{sec}}^+ \phi_r(x)$$

and hence

$$-f(x) \leq \limsup_{r \rightarrow 0} M_{\mathcal{A}_{sec}}^+ \phi_r(x) \leq M_{\mathcal{A}_{sec}}^+ u(x).$$

□

3.4. Continuity of Nonlinear Operators. It is useful to know that general operators like

$$F(u, x) = \inf_{a \in \mathcal{S}} \sup_{K \in \mathcal{A}_a} \left\{ \int_{\mathbb{R}^n} \delta_h^2 u(x) K(h) dh \right\},$$

map $C^{1,1}(\Omega) \cap L^\infty(\mathbb{R}^n) \rightarrow C(\Omega)$, where Ω is an open domain. A proof of this appears in [9], and their proof carries over immediately to our situation of \mathcal{A}_{sec} . Although not exactly stated as such, the result of [9] applies to very general situations. We make no assumptions on $\mathcal{A}_a \subset \mathcal{A}$ other than those stated in the following Lemma:

Lemma 3.6 ([9, Lemma 4.1]). *Assume that for each $r > 0$,*

$$G(h) = \sup_a \sup_{K \in \mathcal{A}_a} (K(h)) \in L^1(\mathbb{R}^n \setminus B_r)$$

(the L^1 norm can depend on r), and that

$$\lim_{s \rightarrow 0} \int_{B_s} |h|^2 G(h) dh = 0.$$

If $\phi \in C^{1,1}(\Omega) \cap L^\infty(\mathbb{R}^n)$, then $F(\phi, \cdot) \in C(\Omega)$.

4. A NONLOCAL REPLACEMENT FOR THE ALEKSANDROV-BAKELMAN-PUCCI ESTIMATE

Pointwise estimates, typically the ABP estimate, are the cornerstone of nonlinear elliptic regularity theory—often they are one of the few places where the equation (4.1) is used. These estimates typically link the supremum of a subsolution with a L^p norm of the right hand side. However in the integro-differential setting, it is well known that such estimates are for the most part a completely open question. When searching for regularity results such as [Theorem 1.1](#), the full strength of the ABP estimate is not necessary, and it suffices to work with a finite cube approximation of $\|f\|_{L^n}$. This was first presented in [9], and has been successfully modified for use in the non-symmetric as well as the parabolic settings ([13], [14], [15]). In this section we provide the necessary modifications to the ABP replacement to treat the larger class of kernels, \mathcal{A}_{sec} .

For this section we assume that u is a subsolution of the equation:

$$\begin{cases} M_{\mathcal{A}_{sec}}^+ u(x) \geq -f(x) & \text{in } B_1 \\ u \leq 0 & \text{on } \mathbb{R}^n \setminus B_1, \end{cases} \quad (4.1)$$

where $f \geq 0$ and $f \in C(\overline{B_1})$. Let $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ be the concave envelope of u^+ in B_3 defined by

$$\Gamma(x) = \begin{cases} \inf\{p(x) : p \text{ is affine and } p \geq u^+ \text{ in } B_3\}, & x \in B_3 \\ 0, & x \in \mathbb{R}^n \setminus B_3. \end{cases} \quad (4.2)$$

Define the contact set in B_1 by $\Sigma = \{u = \Gamma\} \cap B_1$.

The next lemma is the key tool for obtaining the nonlocal replacement for the ABP estimate. It states that for all points in Σ , there is at least one dyadic ring in which u separates sub-quadratically from Γ in a uniformly sized portion of the ring. This is just the right amount

of regularity to eventually show that $\nabla\Gamma$ maps a ball centered at x in the contact set to a uniformly comparable ball in the set of super-differentials. Eventually we conclude with the finite cube replacement for the ABP which appears as Theorem 4.6.

Lemma 4.1. *Let $\rho_0 \in (0, 1)$ and $r_j = \rho_0 2^{-\frac{1}{2-\alpha}-j}$ for $j \in \mathbb{N}_0$. For $x \in \mathbb{R}^n$ define the rings $R_j(x) = B_{r_j}(x) \setminus B_{r_{j+1}}(x)$ and the subsets*

$$R_j(x, V_\xi) = \{z \in R_j(x) : z - x \in V_\xi\} \quad (4.3)$$

$$D_j(x) = \{h \in \mathbb{R}^n : u(x+h) < u(x) + \langle h, \nabla\Gamma(x) \rangle - Ar_j^2\}. \quad (4.4)$$

There exists a constant $C_0 = C_0(n, \delta, \rho_0, \lambda) \geq 1$ such that for every $x \in \Sigma$ and $A > 0$ there is an index $j \in \mathbb{N}_0$ and a conical set V_ξ (ξ depends on x) with the property

$$|R_j(x, V_\xi) \cap \{z \in \mathbb{R}^n : u(z) < u(x) + \langle z - x, \nabla\Gamma(x) \rangle - Ar_j^2\}| \leq C_0 |R_j(x, V_\xi)| \frac{f(x)}{A}. \quad (4.5)$$

Here $\nabla\Gamma(x)$ is any element of the super-differential of Γ in B_3 at x .

Remark. Note that $\nabla\Gamma(x) = \nabla u(x)$ for $x \in \Sigma$ if u is differentiable at x .

Proof of Lemma 4.1. Let $x \in \Sigma$. Since u can be touched by a supporting hyperplane p from above at x , Proposition 3.1 implies that $M_{\mathcal{A}_{sec}}^+ u(x)$ is defined classically and Lemma 3.2 (Remark 3.5) guarantees that there exists a sector, V_ξ (depending on x) such that

$$M_{\mathcal{A}_{sec}}^+ u(x) = (2 - \alpha) \int_{\mathbb{R}^n} (\Lambda(\delta_h^2 u(x))^+ - \lambda(\delta_h^2 u(x))^- \mathbb{1}_{V_\xi}(h)) \mu(dh) \geq -f(x).$$

Note that if both $x+h \in B_3$ and $x-h \in B_3$, then

$$\delta_h^2 u(x) = u(x+h) + u(x-h) - 2u(x) \leq p(x+h) + p(x-h) - 2p(x) = 2p(x) - 2p(x) = 0.$$

Moreover, if either $x+h \notin B_3$ or $x-h \notin B_3$, then $x+h \notin B_1$ and $x-h \notin B_1$. Thus $u(x+h) \leq 0$ and $u(x-h) \leq 0$. Therefore $\delta_h^2 u(x) \leq 0$ for $h \in \mathbb{R}^n$. Thus

$$-f(x) \leq M_{\mathcal{A}_{sec}}^+ u(x) \leq -(2 - \alpha) \int_{B_{r_0}} \lambda \mathbb{1}_{V_\xi}(h) (\delta_h^2 u(x))^- \mu(dh).$$

Recall $r_0 = \rho_0 2^{-1/(2-\alpha)}$. Since $\bigcup_{j=0}^{\infty} R_j(0, V_\xi) \subset B_{r_0}(0)$ and $R_j(0, V_\xi) \cap R_l(0, V_\xi) = \emptyset$ for $j \neq l$, we obtain from the inequality above

$$f(x) \geq (2 - \alpha) \lambda \sum_{j=0}^{\infty} \int_{R_j(0, V_\xi)} \mathbb{1}_{V_\xi}(h) (\delta_h^2 u(x))^- \mu(dh). \quad (4.6)$$

We want to estimate the integrals appearing in (4.6). First note that for $h \in B_1$

$$0 \leq (\delta_h^2 u(x))^- = -\delta_h^2 u(x) = -[u(x+h) - u(x) - \langle h, \nabla\Gamma(x) \rangle] - [u(x-h) - u(x) + \langle h, \nabla\Gamma(x) \rangle].$$

Note that the two terms in the brackets above are nonpositive because of the concavity of Γ . We use the argument from above to estimate each integral in (4.6):

$$\int_{R_j(0, V_\xi)} \mathbb{1}_{V_\xi}(h) (\delta_h^2 u(x))^- \mu(dh) \geq - \int_{R_j(0, V_\xi)} \mathbb{1}_{V_\xi}(h) (u(x+h) - u(x) - \langle h, \nabla\Gamma(x) \rangle) \mu(dh).$$

Let us assume that the assertion of (4.5) fails for all rings, i.e. for every $j \in \mathbb{N}_0$

$$|R_j(x, V_\xi) \cap D_j(x)| > C_0 |R_j(x, V_\xi)| \frac{f(x)}{A}, \quad (4.7)$$

where $D_j(x)$ appears in (4.4). Then

$$\begin{aligned} & - \int_{R_j(0, V_\xi)} \mathbb{1}_{V_\xi}(h)(u(x+h) - u(x) - \langle h, \nabla \Gamma(x) \rangle) \mu(dh) \\ & \geq - \int_{R_j(0, V_\xi) \cap D_j(x)} \mathbb{1}_{V_\xi}(h)(u(x+h) - u(x) - \langle h, \nabla \Gamma(x) \rangle) \mu(dh) \\ & \geq Ar_j^2 \frac{1}{r_j^{n+\alpha}} |R_j(0, V_\xi) \cap D_j(x)| \geq C_0 |R_j(0, V_\xi)| \frac{Af(x)}{Ar_j^{n+\alpha-2}} = c_1 C_0 f(x) r_j^{2-\alpha}, \end{aligned}$$

where $c_1 > 0$ depends on n and δ . Therefore we obtain

$$\begin{aligned} f(x) & \geq c_1(2-\alpha)\lambda C_0 f(x) \sum_{j=0}^{\infty} r_j^{2-\alpha} \\ & = \frac{c_1}{2} \rho_0^{2-\alpha} (2-\alpha)\lambda C_0 f(x) \sum_{j=0}^{\infty} (2^{-(2-\alpha)})^j \\ & \geq \frac{c_1}{2} C_0 f(x) \lambda \rho_0^2 \frac{2-\alpha}{1-2^{-(2-\alpha)}} \geq c_2 C_0 f(x), \end{aligned}$$

with a positive constant c_2 depending on n, δ, ρ_0 and λ . Note that c_2 is independent of α . By choosing C_0 large enough, we obtain a contradiction, and hence (4.5) holds for at least one ring. \square

The goal of the remainder of this section is to construct a specific covering of the contact set $\{u = \Gamma\}$ by a finite number of cubes. We need the following lemma which relates a bound of concave functions in a portion of the ball to an estimate in the whole ball.

Lemma 4.2. *Define $R = B_1 \setminus B_{1/2}$ and $R(0, V_\xi) = R \cap V_\xi$. There exist $l = l(n, \delta) \in (0, \frac{1}{2})$ and $\varepsilon_0(n, \delta) > 0$ such that for every concave function $G : B_1 \rightarrow \mathbb{R}$ and $b > 0$ satisfying*

$$|\{z \in R(0, V_\xi) : G(z) < G(0) + \langle z, \nabla G(0) \rangle - b\}| \leq \varepsilon_0 |R(0, V_\xi)|,$$

the inequality

$$G(y) \geq G(0) + \langle y, \nabla G(0) \rangle - b$$

holds for every $y \in B_l$.

Remark. Note that the assertion of this result is weaker than the corresponding one of [9, Lemma 8.4]. The uniform estimate only capturing $y \in B_l$ is due to the geometric restriction imposed by the set V_ξ . Basically B_l is the largest sized ball contained in the convex hull of $\frac{1}{2}V_\xi \cap B_{1/2}$. See (3.6) for $\frac{1}{2}V_\xi$.

Proof. We will prove this estimate in two steps. First we will show that in half of the sector, $\frac{1}{2}V_\xi$ (see (3.6)), the uniform estimate holds in $B_{1/2}$. Second we note that l can be the radius of the largest ball centered at 0 such that $B_l \subset \text{hull}(B_{1/2} \cap \frac{1}{2}V_\xi)$.

We let $y \in B_{1/2} \cap \frac{1}{2}V_\xi$ be generic. Choose $l_0 \in (0, \frac{1}{2})$ sufficiently small such that one can find two points y_1 and y_2 in $R(0, \frac{1}{2}V_\xi)$ such that

$$y = (y_1 + y_2)/2$$

and both $B_{l_0}(y_1)$ and $B_{l_0}(y_2)$, are contained in $R(0, V_\xi)$. We claim that ε_0 can be chosen small enough, such that for every $y \in B_{1/2} \cap \frac{1}{2}V_\xi$, y_1 and y_2 as above, and every $b > 0$ satisfying

$$|\{z \in R(0, V_\xi) : G(z) < G(0) + \langle z, \nabla G(0) \rangle - b\}| \leq \varepsilon_0 |R(0, V_\xi)|, \quad (4.8)$$

there will be two points $z_1 \in B_{l_0}(y_1)$ and $z_2 \in B_{l_0}(y_2)$ such that

- (i) $y = (z_1 + z_2)/2$,
- (ii) $G(z_1) \geq G(0) + \langle z_1, \nabla G(0) \rangle - b$, and
- (iii) $G(z_2) \geq G(0) + \langle z_2, \nabla G(0) \rangle - b$.

We prove the claim as follows: Choose ε_0 sufficiently small such that

$$\varepsilon_0 |R(0, V_\xi)| < \frac{|B_{l_0}|}{2}.$$

Let $y \in B_{1/2} \cap R(0, V_\xi)$ and $y_1, y_2 \in R(0, V_\xi)$ be arbitrary with $y = (y_1 + y_2)/2$. We define

$$\begin{aligned} D_1 &= \{z_1 \in B_{l_0}(y_1) : G(z_1) \geq G(0) + \langle z_1, \nabla G(0) \rangle - b\} \subset R(0, V_\xi), \\ D_2 &= \{z_2 \in B_{l_0}(y_2) : G(z_2) \geq G(0) + \langle z_2, \nabla G(0) \rangle - b\} \subset R(0, V_\xi). \end{aligned}$$

Using (4.8) and the choice of ε_0 from above, we obtain $|D_1| > \frac{|B_{l_0}|}{2}$ and $|D_2| > \frac{|B_{l_0}|}{2}$. It is clear that for every point $z_1 \in B_{l_0}(y_1)$ there exists a point $z_2 \in B_{l_0}(y_2)$ such that $y = \frac{z_1 + z_2}{2}$. We want to find points $z_1 \in D_1$ and $z_2 \in D_2$ such that $y = \frac{z_1 + z_2}{2}$. Let us assume that this is not possible. Hence, for every $z_1 \in D_1$ we find a point $z_2 \in B_{l_0}(y_2) \setminus D_2$ such that $y = \frac{z_1 + z_2}{2}$. This implies that

$$|B_{l_0}(y_2) \setminus D_2| \geq |D_1| > \frac{|B_{l_0}|}{2}.$$

This is a contradiction to the fact that $|D_2| > \frac{|B_{l_0}|}{2}$. This proves our claim.

For $z_1 \in B_{l_0}(y_1)$ and $z_2 \in B_{l_0}(y_2)$ satisfying (i)-(iii) we finally have

$$\begin{aligned} G(y) &= G\left(\frac{z_1 + z_2}{2}\right) \geq \frac{1}{2}G(z_1) + \frac{1}{2}G(z_2) \\ &\geq G(0) + \frac{1}{2}\langle z_1 + z_2, \nabla G(0) \rangle - b = G(0) + \langle y, \nabla G(0) \rangle - b. \end{aligned}$$

Now to conclude the second step, we simply remark that by concavity the bound must hold for all y in the convex hull of $B_{1/2} \cap \frac{1}{2}V_\xi$. Thus taking l to be the radius of the largest ball contained in the convex hull, we have the estimate for the decay of G for all $y \in B_l$. \square

By a simple scaling argument we get [Lemma 4.2](#) for every ball:

Corollary 4.3. *For $x \in \mathbb{R}^n$ and $r > 0$ define $R_r(x) = B_r(x) \setminus B_{r/2}(x)$ and the subset*

$$R_r(x, V_\xi) = \{y \in R_r(x) : (y - x) \in V_\xi\}.$$

For every concave function $G : B_r(x) \rightarrow \mathbb{R}$ and $b > 0$ satisfying

$$|\{z \in R_r(x, V_\xi) : G(z) < G(x) + \langle z - x, \nabla G(x) \rangle - b\}| \leq \varepsilon_0 |R_r(x, V_\xi)|, \quad (4.9)$$

the inequality

$$G(y) \geq G(x) + \langle y - x, \nabla G(x) \rangle - b$$

holds for every $y \in B_{lr}(x)$, where ε_0 and l are as in [Lemma 4.2](#).

[Corollary 4.3](#) and [Lemma 4.1](#) lead to the following result. The proof is obtained in the same way as in [9, Corollary 8.5]:

Corollary 4.4. *Let $\rho_0 \in (0, 1)$ be arbitrary and ε_0, l be as in [Corollary 4.3](#). There exists a constant $C_1 = C_1(n, \delta, \rho_0, \lambda) \geq 1$ such that for every $x \in \Sigma$ there exist a radius $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ and a sector, V_ξ , (both depending on x) such that*

$$\frac{|\{y \in R_r(x, V_\xi) : u(y) < u(x) + \langle y - x, \nabla \Gamma(x) \rangle - C_1 f(x)(lr)^2\}|}{|R_r(x, V_\xi)|} \leq \varepsilon_0 \quad (4.10)$$

and

$$|\nabla \Gamma(B_{lr/2}(x))| \leq (8C_1)^n f(x)^n |B_{lr/2}(x)|, \quad (4.11)$$

where $R_r(x, V_\xi)$ is defined as in [Corollary 4.3](#).

Proof. Let $x \in \Sigma$ be fixed. Because of [Lemma 4.1](#) there is a constant $C_0 = C_0(n, \delta, \rho_0, \lambda) \geq 1$ and for every $A > 0$ there exists some $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ and a sector V_ξ such that

$$|\{y \in R_r(x, V_\xi) : u(y) < u(x) + \langle y - x, \nabla \Gamma(x) \rangle - Ar^2\}| \leq C_0 \frac{f(x)}{A} |R_r(x, V_\xi)|.$$

By choosing $A = \frac{C_0 f(x)}{\varepsilon_0}$ we obtain [\(4.10\)](#), where $C_1 = \frac{C_0}{\varepsilon_0 l^2}$.

Now let us prove [\(4.11\)](#). First note that for every $b > 0$ the set $\{y \in \mathbb{R}^n : \Gamma(y) < \Gamma(x) + \langle y - x, \nabla \Gamma(x) \rangle - b\}$ is a subset of $\{y \in \mathbb{R}^n : u(y) < u(x) + \langle y - x, \nabla \Gamma(x) \rangle - b\}$. Using this relation and [\(4.10\)](#) we conclude that there is a constant $C_1 = C_1(n, \delta, \rho_0, \lambda) \geq 1$ and some $r \in (0, \rho_0 2^{-1/(2-\alpha)})$ such that

$$\frac{|\{y \in R_r(x, V_\xi) : \Gamma(y) < \Gamma(x) + \langle y - x, \nabla \Gamma(x) \rangle - C_1 f(x)(lr)^2\}|}{|R_r(x, V_\xi)|} \leq \varepsilon_0. \quad (4.12)$$

Because of the concavity of Γ and [\(4.12\)](#), we may apply [Corollary 4.3](#) for $G = \Gamma$ and $b = C_1 f(x)(lr)^2$. We obtain

$$\Gamma(y) \geq \Gamma(x) + \langle y - x, \nabla \Gamma(x) \rangle - C_1 f(x)(lr)^2$$

for every $y \in B_{lr}(x)$. At the same time,

$$\Gamma(y) \leq \Gamma(x) + \langle y - x, \nabla \Gamma(x) \rangle$$

for every $y \in B_{lr}(x)$ because of the concavity of Γ . Hence,

$$|\Gamma(y) - \Gamma(x) - \langle y - x, \nabla \Gamma(x) \rangle| \leq C_1 f(x)(lr)^2 \text{ for every } y \in B_{lr}(x). \quad (4.13)$$

Recall that f is a positive function. [Lemma 4.5\(ii\)](#) – presented below – completes the proof. \square

Lemma 4.5. (i) *Let $G : B_R \rightarrow \mathbb{R}$ be a concave function. Then*

$$\sup_{y \in B_{R/2}} |\nabla G(y)| \leq \frac{4}{R} \sup_{y \in B_R} |G(y)|. \quad (4.14)$$

(ii) *Let $G : B_R \rightarrow \mathbb{R}$ be a concave function satisfying*

$$|G(y) - G(0) - \langle y, \nabla G(0) \rangle| \leq KR^2 \quad (4.15)$$

for every $y \in B_R$ with some $K > 0$. Then

$$|\nabla G(B_{R/2})| \leq (8K)^n |B_{R/2}|. \quad (4.16)$$

Proof. (i) It is sufficient to prove (4.14) for $R = 1$. Set $M = \sup_{y \in B_1} |G(y)|$. Let $y \in B_{1/2}$. Given $h \neq 0$, choose $s < 0 < t$ such that $|y + sh| = |y + th| = 1$. Then

$$-M \leq G(y + sh) \leq G(y) + \langle sh, \nabla G(y) \rangle \leq M + \langle sh, \nabla G(y) \rangle \quad \text{and} \quad |sh| \geq \frac{1}{2}.$$

The same estimates hold when s is replaced by t . Therefore we obtain

$$\langle \nabla G(y), h \rangle \leq -\frac{2M}{s} \leq 4M |h| \quad \text{and} \quad \langle \nabla G(y), h \rangle \geq -\frac{2M}{t} \geq -4M |h|.$$

As a consequence we deduce for every $h \neq 0$ the estimate $\frac{|\langle \nabla G(y), h \rangle|}{|h|} \leq 4M$. Hence we obtain $|\nabla G(y)| \leq 4M$, which finishes the proof of (4.14).

(ii) For $y \in B_R$ define $\widehat{G}(y) = G(y) - G(0) - \langle y, \nabla G(0) \rangle$. \widehat{G} is a concave function in B_R . Let $z \in B_{R/2}$. Using (4.14) and (4.15) we obtain

$$|\nabla G(z) - \nabla G(0)| = |\nabla \widehat{G}(z)| \leq \frac{4}{R} \sup_{y \in B_R} |G(y) - G(0) - \langle y, \nabla G(0) \rangle| \leq 4KR = 8K \frac{R}{2}.$$

Therefore

$$\nabla G(B_{R/2}) \subset B_{8K(R/2)}(\nabla G(0)) \quad \text{and} \quad |\nabla G(B_{R/2})| \leq (8K)^n |B_{R/2}|. \quad \square$$

As a consequence of Corollary 4.4 we derive a theorem which can be considered as a nonlocal finite cube substitute for the classical ABP estimate, cf. [12, Theorem 3.2] and Corollary 4.7.

Theorem 4.6. *Let $l \in (0, \frac{1}{2})$ be as in Corollary 4.4 and assume $0 < \rho_0 \leq \frac{l}{16n}$. There exist constants $C_2 = C_2(\delta, \lambda, \rho_0, n) \geq 1$ and $\nu = \nu(\delta, n) > 0$ such that for any u solving (4.1) there exists a disjoint family of open cubes $(Q^j)_{j=1, \dots, m}$, $m \in \mathbb{N}$, with diameters $0 < d_j \leq \rho_0 2^{-1/(2-\alpha)}$ which covers the contact set $\Sigma = \{u = \Gamma\} \cap B_1$ such that the following properties hold for every $j = 1, \dots, m$:*

- (1) $\Sigma \cap \overline{Q^j} \neq \emptyset$.
- (2) $|\nabla \Gamma(\overline{Q^j})| \leq C_2 (\sup_{Q^j} f)^n |Q^j|$.
- (3) $|\{y \in \eta Q^j : u(y) \geq \Gamma(y) - C_2 (\sup_{Q^j} f) d_j^2\}| \geq \nu |\eta Q^j|$, where $\eta = (1 + \frac{8}{l})\sqrt{n}$.

Proof. The proof follows the one of [9, Theorem 8.7]. In our context, the main constants additionally depend on δ . Let $C_1 = C_1(n, \delta, \rho_0, \lambda) \geq 1$ be as in Corollary 4.4. Set $c_1 = (8C_1)^n$ and $c_2 = 16C_1$. We prove the assertion of the theorem with $C_2 = c_1 \eta^n$ and $\nu = (1 - l) \frac{|R(0, V_\xi)|}{|B_1|} (8\sqrt{n})^{-n}$, where $R(0, V_\xi)$ is as in Lemma 4.2.

Let \mathcal{Q}_1 be a finite disjoint family of open cubes Q with diameter $d_1 = \rho_0 2^{-1/(2-\alpha)}$ and the property $B_1 \subset \bigcup_{\mathcal{Q}_1} \overline{Q}$. Let $\mathcal{Q}'_1 \subset \mathcal{Q}_1$ be the subfamily of all cubes Q with $\overline{Q} \cap \Sigma \neq \emptyset$. We decompose every cube in \mathcal{Q}'_1 which does not satisfy both conditions (2) and (3) from above into 2^n sub-cubes with half diameter. Now, let \mathcal{Q}_2 be the family of these newly created sub-cubes plus those cubes from \mathcal{Q}'_1 that do satisfy both conditions (2) and (3) from above (and hence were not decomposed). We repeat this procedure and obtain a sequence of families

$$\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \dots$$

We claim that there is an index $k \in \mathbb{N}$ with $\mathcal{Q}_k = \mathcal{Q}_{k+i}$ for all $i \in \mathbb{N}$. In this case, we set $m = \#\mathcal{Q}_k$.

Let us assume that no such index $k \in \mathbb{N}$ exists. Then there exists a sequence of cubes Q^j with diameter d_j such that $d_j = 2^{-j+1}d_1$ and for every $j \in \mathbb{N}$ the following conditions hold:

- $Q^j \supset Q^{j+1}$.
- $\overline{Q^j} \cap \Sigma \neq \emptyset$.
- Q^j violates (2) or (3).

Let $x_0 \in \mathbb{R}^n$ satisfy $\{x_0\} = \bigcap_{j \in \mathbb{N}} \overline{Q^j}$. Firstly, we claim $x_0 \in \Sigma = \{u = \Gamma\} \cap B_1$. It is sufficient to prove $x_0 \in \{u = \Gamma\}$. Note that there is a sequence $(x_j)_{j \in \mathbb{N}}$ with $x_0 = \lim_{j \rightarrow \infty} x_j$ and $x_j \in \overline{Q^j} \cap \Sigma$ for every $j \in \mathbb{N}$, and hence $x_0 \in \Sigma$ since Σ is closed.

We now derive a contradiction by showing that one of the cubes Q^j from above satisfies (2) and (3). Using [Corollary 4.4](#), there is a number r with $0 < r < \rho_0 2^{-1/(2-\alpha)}$ such that

$$\frac{|\{y \in R(x_0, V_\xi) : u(y) < u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle - C_1 f(x_0)(lr)^2\}|}{|R(x_0, V_\xi)|} \leq l \quad (4.17)$$

and

$$|\nabla \Gamma(B_{lr/2}(x_0))| \leq c_1 f(x_0)^n |B_{lr/2}(x_0)|. \quad (4.18)$$

Fix an index $j_0 \in \mathbb{N}$ such that $\frac{lr}{4} \leq d_{j_0} < \frac{lr}{2}$. Therefore

$$B_{lr/2}(x_0) \supset \overline{Q^{j_0}}, \quad B_r(x_0) \subset \eta Q^{j_0} \subset B_3. \quad (4.19)$$

Note that $\Gamma(y) \leq u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle$ in B_3 . Recall $\eta = (1 + \frac{8}{l})\sqrt{n}$. Using (4.17), (4.19) and the relation between d_{j_0} and r , we obtain

$$\begin{aligned} & |\{y \in \eta Q^{j_0} : u(y) \geq \Gamma(y) - C_2(\sup_{\overline{Q^{j_0}}} f)d_{j_0}^2\}| \\ & \geq |\{y \in \eta Q^{j_0} : u(y) \geq u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle - c_2 f(x_0)\frac{(lr)^2}{16}\}| \\ & \geq |\{y \in R(x_0, V_\xi) : u(y) \geq u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle - C_1 f(x_0)(lr)^2\}| \\ & \geq |R(x_0, V_\xi)| - l |R(x_0, V_\xi)| = (1 - l) |R(x_0, V_\xi)| \geq \nu |\eta Q^{j_0}|. \end{aligned}$$

Moreover, using (4.18) and (4.19), we obtain

$$|\nabla \Gamma(\overline{Q^{j_0}})| \leq |\nabla \Gamma(B_{lr/2}(x_0))| \leq c_1 f(x_0)^n |B_{lr/2}(x_0)| \leq c_1 (\sup_{\overline{Q^{j_0}}} f)^n |\eta Q^{j_0}| = C_2 (\sup_{\overline{Q^{j_0}}} f)^n |Q^{j_0}|.$$

Therefore Q^{j_0} satisfies (1) – (3) with C_2, ν from above. Contradiction. \square

The following corollary can be seen as a discretized version of the Aleksandrov-Bakelman-Pucci estimate [12, Theorem 3.2] in our setting. Note that the index m in the assertion below depends on α with $m \rightarrow +\infty$ for $\alpha \rightarrow 2-$.

Corollary 4.7. *Under the assumptions of [Theorem 4.6](#) we have*

$$\sup_{B_1} u^+ \leq C_3 \left(\sum_{j=1}^m (\sup_{\overline{Q^j}} f)^n |Q^j| \right)^{1/n},$$

with $m \in \mathbb{N}$, (Q^j) as in [Theorem 4.6](#) and $C_3 = C_3(n, \delta, \rho_0, \lambda) \geq 1$.

Proof. Set $S = \sup_{B_1} u^+$. Since $u^+ = 0$ in $\mathbb{R}^n \setminus B_1$ and u is upper semicontinuous, there is $x_0 \in B_1$ with $S = u^+(x_0)$. Using the geometric argument given in the proof of [17, Lemma 9.2] we deduce

$$B_{S/4} \subset \nabla\Gamma(B_1) \quad \Rightarrow \quad S^n \leq c_1 |\nabla\Gamma(B_1)|,$$

with some constant $c_1 = c_1(n) \geq 1$. Part (2) of [Theorem 4.6](#) now implies

$$\sup_{B_1} u^+ \leq c_1^{1/n} |\nabla\Gamma(B_1)|^{1/n} = c_1^{1/n} |\nabla\Gamma(B_1 \cap \{u = \Gamma\})|^{1/n} \leq C_3 \left(\sum_{j=1}^m (\sup_{Q^j} f)^n |Q^j| \right)^{1/n},$$

where $C_3 \geq 1$ depends only on n, δ, ρ_0 and λ . Here, we have used the fact that

$$\nabla\Gamma(B_1 \cap \{u = \Gamma\}) = \nabla\Gamma(B_1), \quad (4.20)$$

which follows from (4.2) and the properties $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and $u^+ \not\equiv 0$. \square

5. A SPECIAL BUMP FUNCTION

In this section we construct a special function with the properties as the one in [12, Lemma 4.1]. We will use this function in Section 6 in combination with the ABP substitute from the previous section. The construction is based on an idea used in [9], but differs significantly to deal with the fact that the mass of the kernels, $K(y)$, could be concentrated on only a small sector, V_ξ . This special function will appear at the end of this section in [Corollary 5.5](#).

To begin, we will consider a two parameter family of functions $f_{\gamma,p} \in C^{1,1}(\mathbb{R}^n)$ given by

$$f_{\gamma,p}(y) = \hat{f}(|y|)$$

and

$$\hat{f}(r) = \begin{cases} r^{-p} & \text{if } r \geq 1 - \frac{C_4}{2} \\ m_{\gamma,p}(r) & \text{if } 1 - C_4 \leq r \leq 1 - \frac{C_4}{2} \\ \gamma^{-p} & \text{if } r \leq 1 - C_4. \end{cases} \quad (5.1)$$

Here we choose to take the middle function, $m_{\gamma,p}$, so that \hat{f} is in $C^{1,1}(\mathbb{R}^n)$ and monotone decreasing for $r \in [0, \infty)$. The value of $C_4 \in (0, 1)$ depends only on δ via the opening of the sectors, V_ξ , and is chosen so that for some universal $\mu_0 > 0$

$$\text{for all } \xi \in \mathbb{S}^{n-1}, |\{y \in V_\xi : e_1 + y \in B_{1-C_4}(0)\}| \geq \mu_0. \quad (5.2)$$

We note that in order to actually attain a monotone function from this construction, we will take $\gamma \leq \gamma_0 = 1/4$ and $C_4 < 1/2$.

The special function will be constructed in two phases. First in [Lemma 5.3](#) we will find a value of p large enough that we can make $M_{\mathcal{A}_{sec}}^- f_{\gamma,p} \geq 0$ for all α near 2. Then in [Lemma 5.4](#) we take γ small enough to cover the range of α down to α_0 . Before we get to those results, we note a couple of useful properties of the family $\{f_{\gamma,p}\}$.

Note 5.1. If $\gamma_1 < \gamma_2$ and p is fixed, then for all y

$$f_{\gamma_1,p}(y) \geq f_{\gamma_2,p}(y),$$

and the two functions are equal when $|y| \geq 1 - \frac{C_4}{2}$, hence

$$M_{\mathcal{A}_{sec}}^- f_{\gamma_1,p}(x) \geq M_{\mathcal{A}_{sec}}^- f_{\gamma_2,p}(x),$$

for all $|x| \geq 1 - \frac{C_4}{2}$.

We also record a useful reduction for the computations.

Lemma 5.2 ([9, p.623]). *Let $f = f_{\gamma,p}$. If $M_{\mathcal{A}_{sec}}^- f(e_1) \geq 0$, then $M_{\mathcal{A}_{sec}}^- f(x) \geq 0$ for all $|x| \geq 1$.*

Proof. First, we note that $M_{\mathcal{A}_{sec}}^-$ is rotationally invariant due to the definition of \mathcal{A}_{sec} (it could fail to be rotationally invariant if \mathcal{A}_{sec} was a smaller collection of kernels). Therefore by the radial symmetry of f we see that

$$M_{\mathcal{A}_{sec}}^- f(x) = M_{\mathcal{A}_{sec}}^- f(|x| e_1)$$

for all $x \in \mathbb{R}^n$ for which $M_{\mathcal{A}_{sec}}^-$ is well defined.

Second, we use [Note 5.1](#) to reduce the calculation to the lowest value of γ , say γ_0 which will be fixed below (in fact $\gamma_0 = 1$ will suffice). Indeed, assuming we have proved that

$$M_{\mathcal{A}_{sec}}^- f_{\gamma_0,p}(x) \geq 0,$$

then [note 5.1](#) gives that for all $\gamma < \gamma_0$

$$M_{\mathcal{A}_{sec}}^- f_{\gamma,p}(x) \geq M_{\mathcal{A}_{sec}}^- f_{\gamma_0,p}(x).$$

Third we see that if

$$\tilde{f}(x) = c^p f(cx),$$

then whenever $c > 1$ and $|x| > 1 - \frac{C_4}{2}$

$$\tilde{f}(x) = c^p |cx|^{-p} = f(x),$$

and one can check that $\tilde{f}(x) \geq f(x)$ for all x .

To conclude, let $x_0 = |x_0| e_1$ be fixed, and let $c = |x_0|$. Then we note that (using $f(x) = c^{-p} \tilde{f}(\frac{x}{c})$)

$$\begin{aligned} M_{\mathcal{A}_{sec}}^- f_{\gamma_0,p}(x_0) &= M_{\mathcal{A}_{sec}}^- c^{-p} \tilde{f}\left(\frac{\cdot}{c}\right)(x_0) \\ &= c^{-p} c^{-\alpha} M_{\mathcal{A}_{sec}}^- \tilde{f}\left(\frac{x_0}{c}\right) \\ &\geq c^{-p-\alpha} M_{\mathcal{A}_{sec}}^- f\left(\frac{x_0}{c}\right) \\ &= c^{-p-\alpha} M_{\mathcal{A}_{sec}}^- f(e_1) \\ &\geq 0 \text{ (by assumption).} \end{aligned}$$

□

Lemma 5.3. *Let $\gamma_0 = 1$ be fixed. Then, there exist a p_0 and an α_1 , depending only on $\gamma_0, C_0, \delta, n, \lambda, \Lambda$, such that*

$$M_{\mathcal{A}_{sec}}^- f_{\gamma_0,p}(x) \geq 0 \quad \text{for all } |x| > 1,$$

for all orders, $\alpha \in (\alpha_1, 2)$.

Proof. We first note that by [Lemma 5.2](#), it suffices to estimate only $M_{\mathcal{A}_{sec}}^- f(e_1)$.

Let K be any kernel in \mathcal{A}_{sec} . In the end of the proof, no constants will depend on this particular K . Let $\mathbb{1}_\xi$ be its corresponding lower bound sector. Let us drop the parameters γ, p from f for

ease of notation. We will first include some preliminary calculations and choices of constants. Then at the end, we put all of these calculations together to conclude the lemma.

For this part of the construction of the special function, α will be close to 2, and therefore it is the local behavior of f which is essential. We start by focusing on the contribution to $M_{\mathcal{A}_{sec}}^- f(e_1)$ given by the integration for $h \in B_r$. We make an important emphasis that for each γ, p, α there is a direction which optimizes $M_{\mathcal{A}_{sec}}^- f(e_1)$ which depends on all of γ, p , and α via (3.8). Let us call that direction ξ throughout, but with the understanding that it depends on γ, p, α . None of the estimates we prove will depend at all on the specific choice of ξ — they will only depend on the opening of the sector, δ , and the other universal parameters.

Whenever $r < \frac{C_4}{2}$, then the inequality ([9, p.624]) holds

$$\delta_h^2 f(e_1) \geq p \left((-|h|^2 + (p+2)(h_1)^2 - \frac{1}{2}(p+2)(p+4)(h_1)^2 |h|^2) \right). \quad (5.3)$$

Therefore we fix let $r_0 = \frac{C_4}{2}$ for the remaining calculations. Next we observe if $h \in \mathbb{R}^n \setminus B_{r_0}$, then

$$\delta_h^2 f(e_1) \geq 2(\inf(f) - f(e_1)) = -2. \quad (5.4)$$

We can thus conclude a bound from below on the contribution outside of B_{r_0} ,

$$\begin{aligned} (2-\alpha) \int_{\mathbb{R}^n \setminus B_{r_0}} -\Lambda(\delta_h^2 f(e_1))^+ + \lambda \mathbb{1}_{V_\xi}(\delta_h^2 f(e_1))^- \mu(dh) &\geq (2-\alpha) \int_{\mathbb{R}^n \setminus B_{r_0}} -\Lambda(\delta_h^2 f(e_1))^- \mu(dh) \\ &\geq (-2)(2-\alpha) \int_{\mathbb{R}^n \setminus B_{r_0}} \Lambda \mu(dh). \end{aligned} \quad (5.5)$$

In order to determine the good value of p_0 , we note first there exists a μ_1 which depends only on n and the sector opening, δ , such that for all ξ

$$\int_{\partial B_1} (z_1)^2 \mathbb{1}_{V_\xi}(z) dS(z) \geq \mu_1. \quad (5.6)$$

Choose p_0 large enough so that

$$\lambda(p_0+2) \int_{\partial B_1} (z_1)^2 \mathbb{1}_{V_\xi}(z) dS(z) - \Lambda \int_{\partial B_1} |z|^2 dS(z) \geq C_5 > 0. \quad (5.7)$$

We point out that this choice of p_0 (and hence C_5) depends only on $\mu_1, n, \lambda, \Lambda$, and since μ_1 depends only on n and δ , p_0 depends on those as well. It is essential to note that p_0 does not depend on γ_0 .

Now we can estimate the contribution to $M_{\mathcal{A}_{sec}}^- f(e_1)$ from B_{r_0} . We note that we only care about $(\delta_y^2 f(e_1))^+$. This is one place where the original definition of $M_{\mathcal{A}_{sec}}^-$ in (3.5) is more helpful than the formula from Lemma 3.8. First we use (5.3) then (1.2), and finally (5.7).

$$\begin{aligned} \int_{B_{r_0}} (\delta_h^2 f(e_1)) K(h) dh &= \int_0^{r_0} s^{n-1} \int_{\partial B_1} (\delta_{sz}^2 f(e_1)) K(sz) dS(z) ds \\ &\geq \int_0^{r_0} s^{n-1} \int_{\partial B_1} p_0 \left((-|sz|^2 + (p_0+2)(sz_1)^2 - \frac{1}{2}(p_0+2)(p_0+4)(sz_1)^2 |sz|^2) \right) K(sz) dS(z) ds \\ &\geq (2-\alpha) \int_0^{r_0} s^2 s^{-1-\alpha} \left[\int_{\partial B_1} \lambda p_0 (p_0+2) (z_1)^2 \mathbb{1}_{V_\xi}(z) dS(z) - \int_{\partial B_1} \Lambda p_0 |z|^2 dS(z) \right] \end{aligned}$$

$$\begin{aligned}
& -s^2 \int_{\partial B_1} \Lambda \frac{1}{2} p_0(p_0+2)(p_0+4)(z_1)^2 |z|^2 dS(z) \Big] ds \\
& \geq (2-\alpha) \int_0^{r_0} s^2 s^{-1-\alpha} [p_0 C_5 - s^2 \omega(n) \Lambda \frac{1}{2} p_0(p_0+2)(p_0+4)] ds \\
& = \lambda p_0 C_5 (r_0)^{2-\alpha} - \omega(n) \Lambda \frac{1}{2} \frac{2-\alpha}{4-\alpha} p_0(p_0+2)(p_0+4) (r_0)^{4-\alpha}.
\end{aligned} \tag{5.8}$$

And we put all the pieces together with (5.8) and (5.4)

$$\begin{aligned}
L_K(f, e_1) &= (2-\alpha) \int_{\mathbb{R}^n} \delta_h^2 f(e_1) K(h) dh \\
&= \int_{B_{r_0}} \delta_h^2 f(e_1) K(h) dh + \int_{\mathbb{R}^n \setminus B_{r_0}} \delta_h^2 f(e_1) K(h) dh \\
&\geq \lambda p_0 C_5 (r_0)^{2-\alpha} - \omega(n) \Lambda \frac{1}{2} \frac{2-\alpha}{4-\alpha} p_0(p_0+2)(p_0+4) (r_0)^{4-\alpha} - 2(2-\alpha) \int_{\mathbb{R}^n \setminus B_{r_0}} \Lambda \mu(dh)
\end{aligned} \tag{5.9}$$

To conclude, we take α_1 close enough to 2 so that for all $\alpha \in (\alpha_1, 2)$, (5.9) is ≥ 0 . Hence by the definition of $M_{\mathcal{A}_{sec}}^-$, it follows that $M_{\mathcal{A}_{sec}}^- f(e_1) \geq 0$, which concludes the lemma. \square

Now that we have the behavior of the special function controlled for $\alpha \in (\alpha_1, 2)$, we need to get the behavior for $\alpha \in (\alpha_0, \alpha_1]$.

Lemma 5.4. *Let α_0 be the lower bound on α given in the introduction. Let α_1 and p_0 be fixed from Lemma 5.3. Then there exists $\gamma_1 > 0$, depending only on C_4, μ_0 (and hence δ) $\alpha_0, \alpha_1, p_0, n, \lambda, \Lambda$, such that for all $\alpha \in (\alpha_0, 2)$ and $|x| > 1$*

$$M_{\mathcal{A}_{sec}}^- f_{\gamma_1, p_0}(x) \geq 0. \tag{5.10}$$

Proof. Recall we will use $\gamma_0 = 1/4$. There are some lower bounds which will be much easier to treat if we have one function to plug into the integrals for all choices of γ . To this end, we introduce an auxiliary function simply for the sake of some estimates. Let $\phi_{\gamma, p}$ be

$$\phi_{\gamma, p}(y) = \min\{\gamma^{-p}, |y|^{-p}\}. \tag{5.11}$$

We know that $\phi_{\gamma, p} \in C^2(\mathbb{R}^n \setminus B_\gamma)$.

Furthermore, we can use the function ϕ_{γ, p_0} as a lower bound for f_{γ, p_0} . Indeed, let $\gamma \leq \gamma_0$. Then we have that $\phi_{\gamma, p_0}(y) = f_{\gamma, p_0}(y)$ whenever $|y| \geq 1 - \frac{C_4}{2}$, and also that $\phi_{\gamma, p_0}(y) \leq f_{\gamma, p_0}(y)$ for all y . Also for $|x| \geq 1$,

$$M^- \phi_{\gamma, p_0}(x) \geq -C \|\phi_{\gamma, p_0}\|_{C^{1,1}(\mathbb{R}^n \setminus B_{1/2})}. \tag{5.12}$$

Finally we note that the $C^{1,1}(\mathbb{R}^n \setminus B_{1/2})$ norm of ϕ_{γ, p_0} is independent of γ .

Using ϕ_{γ_0, p_0} we see that there exists a C depending only on n, λ, Λ such that

$$\Lambda \int_{\mathbb{R}^n} -(\delta_y^2 f_{\gamma_0, p_0}(e_1))^- |y|^{-n-\alpha} dy \geq -C(\|\phi_{\gamma_0, p_0}\|_{C^{1,1}(B_{1/2}(e_1))} + 2(1 - \inf_{\mathbb{R}^n}(f))), \tag{5.13}$$

for all $\gamma \leq \gamma_0$.

We now proceed with the calculation which will lead to the choice of γ_1 . Just as in the previous lemma, we work with L_K instead of M^- , and the result follows because no estimates depend on the particular choice of K .

We drop the γ, p_0 subscripts until the end.

$$\begin{aligned}
& \int_{\mathbb{R}^n} \delta_h^2 f(e_1) K(h) dh \\
& \geq (2 - \alpha) \left[\lambda \int_{\mathbb{R}^n} (\delta_h^2 f(e_1))^+ \mathbb{1}_{V_\xi}(h) |h|^{-n-\alpha} dh - \Lambda \int_{\mathbb{R}^n} (\delta_h^2 f(e_1))^- |h|^{-n-\alpha} dh \right] \\
& \geq (2 - \alpha) \lambda \int_{\{y : e_1 \pm y \in B_{1-C_4}(0)\}} (\delta_h^2 f(e_1))^+ \mathbb{1}_{V_\xi}(h) |h|^{-n-\alpha} dh \\
& \quad - (2 - \alpha) \Lambda C(n, \alpha_0) (\|\phi_{\gamma_0, p_0}\|_{C^{1,1}(B_{1/2}(e_1))} + 2(1 - \inf_{\mathbb{R}^n} f)) \\
& \geq (2 - \alpha) \lambda \int_{\{y : e_1 \pm y \in B_{1-C_4}(0)\}} (\gamma^{-p_0} - 1) |h|^{-n-\alpha} dh \\
& \quad - (2 - \alpha) \Lambda C(n, \alpha_0) (\|\phi_{\gamma_0, p_0}\|_{C^{1,1}(B_{1/2}(e_1))} + 2) \\
& \geq (2 - \alpha) \lambda (2 - C_4)^{-n-\alpha} (\gamma^{-p_0} - 1) |\{y : e_1 \pm y \in B_{1-C_4}(0)\}| \\
& \quad - (2 - \alpha) \Lambda C(n, \alpha_0) (\|\phi_{\gamma_0, p_0}\|_{C^{1,1}(B_{1/2}(e_1))} + 2) \\
& \geq (2 - \alpha) \lambda (2 - C_4)^{-n-\alpha} (\gamma^{-p_0} - 1) \mu_0 - (2 - \alpha) \Lambda C(n, \alpha_0) (\|\phi_{\gamma_0, p_0}\|_{C^{1,1}(B_{1/2}(e_1))} + 2)
\end{aligned}$$

We note that if $h \in \{y : e_1 \pm y \in B_{1-C_4}(0)\}$, then $C_4 \leq |h| \leq 2 - C_4$, and hence $|h|^{-n-\alpha} \geq (2 - C_4)^{-n-\alpha}$. Also we note $\inf(f) = 0$. Now γ_1 can be chosen to depend on only α_1, p_0, μ_0 (and hence δ), n, λ, Λ , so that the final line becomes ≥ 0 . \square

Corollary 5.5. *Assume $0 < R \leq 1$ is given. There exists a continuous function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ (depending on R) with the following properties:*

- (1) $\Phi(x) = 0$ for every $x \in \mathbb{R}^n \setminus B_{2\sqrt{n}}$.
- (2) $\Phi(x) > 2$ for every $x \in Q_3$.
- (3) There exists a bounded, nonnegative function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ (depending on R), supported in $\overline{B_R}$, such that $M_{\mathcal{A}_{sec}}^- \Phi(x) \geq -\psi(x)$ for every $x \in \mathbb{R}^n$ and every $\alpha \in (\alpha_0, 2)$.

Proof. We begin by noting that by construction, f_{γ_0, p_0} is $C^{1,1}(\mathbb{R}^n)$. We therefore have that given any $0 < R < 1$ fixed, the function $f_R(x) := f(\frac{x}{R})$ satisfies classically by a rescaling of $M_{\mathcal{A}_{sec}}^- f(x) \geq 0$ in $\{|x| \geq 1\}$,

$$M_{\mathcal{A}_{sec}}^- f_R(x) = R^{-\alpha} M_{\mathcal{A}_{sec}}^- f\left(\frac{x}{R}\right) \geq 0 \quad \text{whenever } |x| \geq R. \quad (5.14)$$

Next we can subtract the constant $(2\sqrt{n})^{-p_0}$. Since constants are subsolutions of $M_{\mathcal{A}_{sec}}^- \geq 0$ and the maximum of two subsolutions is still a subsolution, we see that

$$\tilde{f}_R = \max\{f_R - (2\sqrt{n})^{-p_0}, 0\} \quad (5.15)$$

still satisfies $M_{\mathcal{A}_{sec}}^- \tilde{f}_R \geq 0$ in the set $|x| \geq R$.

Finally to conclude, we choose c large enough so that

$$\Phi(x) := c \max\{f_R(x) - (2\sqrt{n})^{-p_0}, 0\} > 2 \quad \text{for all } x \in Q_3. \quad (5.16)$$

To conclude we comment that by [Lemma 3.6](#), $M_{\mathcal{A}_{sec}}^- \Phi$ is continuous in \mathbb{R}^n , and hence defining $\psi = M_{\mathcal{A}_{sec}}^+ \Phi$, we conclude. \square

6. POINT TO MEASURE ESTIMATES AND HÖLDER REGULARITY

This section contains the main auxiliary result, [Lemma 6.1](#), which is the key to [Theorem 1.1](#). The proof of [Lemma 6.1](#) uses the main contributions of this article, [Theorem 4.6](#) and [Corollary 5.5](#). Once [Lemma 6.1](#) is established, a-priori Hölder regularity estimates follow by the classical method of oscillation reduction.

Lemma 6.1. *There exist constants $\varepsilon_0 > 0$, $\kappa \in (0, 1)$ and $A > 1$ (depending only on $\lambda, \Lambda, n, \delta$ and α_0) such that for every $\alpha \in (\alpha_0, 2)$ and every bounded function $w : \mathbb{R}^n \rightarrow \mathbb{R}$ which is lower semicontinuous in $\overline{Q_{4\sqrt{n}}}$ and satisfies*

- (1) $w \geq 0$ in \mathbb{R}^n ,
- (2) $\inf_{Q_3} w \leq 1$, and
- (3) $M_{A_{sec}}^- w \leq \varepsilon_0$ in $Q_{4\sqrt{n}}$ in the viscosity sense,

we have

$$|\{w \leq A\} \cap Q_1| \geq \kappa. \quad (6.1)$$

Proof. The proof uses the same strategy as the one of [9, Lemma 10.1]. Here, the size of the cubes used in the localization argument depends on δ . Let $l \in (0, \frac{1}{2})$ be as in [Corollary 4.3](#). Set $R = \frac{l}{8}$ and $u = \Phi - w$, where Φ is the special function constructed in [Corollary 5.5](#). Let us summarize properties of u .

- u is upper semicontinuous in $\overline{B_{2\sqrt{n}}}$,
- $u \leq 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$,
- For every $\alpha \in (\alpha_0, 2)$ $M_\alpha^+ u \geq M_\alpha^- \Phi - M_\alpha^- w \geq -\psi - \varepsilon_0$ in $Q_{4\sqrt{n}} \supset B_{2\sqrt{n}}$ in the viscosity sense, where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is as in [Corollary 5.5](#).

Let Γ be the concave envelope of u^+ in $B_{6\sqrt{n}}$. Next, we apply rescaled versions of [Theorem 4.6](#) and [Corollary 4.7](#) with $\rho_0 = 2\sqrt{n} \frac{l}{16n} = \frac{l}{8\sqrt{n}}$. Note that [Theorem 4.6](#) is formulated for subsolutions in B_1 . Using a scaling argument, the assertion remains true when considering subsolutions in $B_{2\sqrt{n}}$ as here. Let $(Q^j)_{j=1, \dots, m}$ be the family of cubes in the rescaled version of [Theorem 4.6](#), i.e. with diameters $d_j \leq \frac{l}{8\sqrt{n}} 2^{-1/(2-\alpha)}$. From [Corollary 4.7](#) we conclude

$$\sup_{B_{2\sqrt{n}}} u \leq C_3 \left(\sum_{j=1}^m (\sup_{Q^j} \psi + \varepsilon_0)^n |Q^j| \right)^{1/n} \leq c_1 \varepsilon_0 + c_1 \left(\sum_{j=1}^m (\sup_{Q^j} \psi)^n |Q^j| \right)^{1/n},$$

where $c_1 = c_1(\lambda, \delta, n) \geq C_3$. The properties $\inf_{Q_3} w \leq 1$ and $\Phi > 2$ in Q_3 imply $\sup_{B_{2\sqrt{n}}} u \geq 1$. Set $\varepsilon_0 = \frac{1}{2c_1}$. Since ψ is supported in $\overline{B_R}$, we obtain

$$\frac{1}{2} \leq c_1 \sup_{B_R} |\psi| \left(\sum_{\substack{j=1, \dots, m \\ Q^j \cap B_R \neq \emptyset}} |Q^j| \right)^{1/n}.$$

Hence there is $c_2 = c_2(\delta, \lambda, \Lambda, \alpha_0, n) > 0$ such that

$$\sum_{\substack{j=1, \dots, m \\ Q^j \cap B_R \neq \emptyset}} |Q^j| \geq c_2. \quad (6.2)$$

Set $c_3 = C_2(\sup_{\mathbb{R}^n}(\psi + \varepsilon_0))$ with C_2 from [Theorem 4.6](#) which we now apply: There is $\nu > 0$ such that for every $j = 1, \dots, m$

$$|\{y \in \eta Q^j : u(y) \geq \Gamma(y) - c_3 d_j^2\}| \geq \nu |\eta Q^j|. \quad (6.3)$$

Let us consider the family

$$\mathcal{Q} = \{\eta Q^j : Q^j \cap B_R \neq \emptyset, j \in \{1, \dots, m\}\},$$

which is an open covering of the set

$$U = \bigcup_{\substack{j=1, \dots, m \\ Q^j \cap B_R \neq \emptyset}} \overline{Q^j}.$$

Given a finite family $\mathcal{A} = \{A_i : 1 \leq i \leq N\}$ of sets $A_i \subset \mathbb{R}^n$ we define the overlapping number by $\omega(\mathcal{A}) = \max_{1 \leq i \leq N} \#\{A \in \mathcal{A} : A \cap A_i \neq \emptyset\}$. In general, $\omega(\mathcal{Q})$ depends on m . However, by a simple covering argument there is a subfamily \mathcal{Q}' of \mathcal{Q} that still covers U but with $\omega(\mathcal{Q}')$ independent of m .

Note that the diameters d_j of the cubes Q^j satisfy $d_j \leq \frac{l}{8\sqrt{2n}}$. Furthermore, $Q^j \cap B_R \neq \emptyset$ implies $\eta Q^j \subset B_{1/2}$ due to the choice of ρ_0 and because of $\eta = (1 + \frac{8}{l})\sqrt{n}$. Thus it follows from [\(6.2\)](#) and [\(6.3\)](#) that

$$|\{y \in B_{1/2} : u(y) \geq \Gamma(y) - c_3 \rho_0^2\}| \geq \kappa, \quad (6.4)$$

where $\kappa \in (0, 1)$ only depends on $\Lambda, \lambda, \delta, \alpha_0$ and n . Let $A_0 = \sup_{B_{1/2}} \Phi$. Since for $y \in B_{1/2}$ $u(y) \geq \Gamma(y) - c_3 \rho_0^2$ implies $w(y) \leq A_0 + c_3 \rho_0^2$, we obtain from [\(6.4\)](#)

$$|\{y \in B_{1/2} : w(y) \leq A_0 + c_3 \rho_0^2\}| \geq \kappa.$$

Set $A = A_0 + c_3 \rho_0^2$. Since $B_{1/2} \subset Q_1$, we have

$$|\{y \in Q_1 : w(y) \leq A\}| \geq \kappa,$$

which finishes the proof. \square

The typical presentation of Hölder regularity from the point-to-measure estimates proceeds via the oscillation reduction lemma of De Giorgi, which actually uses the Harnack inequality (see [\[12, Chapter 4\]](#), [\[17, Chapters 8, 9\]](#)). However as mentioned in [Section 2](#), the Harnack inequality fails for operators in the class \mathcal{A}_{sec} , and so some care must be used. That is why we choose to cite the direct methods presented in [\[9, Section 12\]](#). There it is directly shown how [Lemma 6.1](#) implies [Theorem 1.1](#). Furthermore once [Lemma 6.1](#) is established, the particular properties of \mathcal{A}_{sec} are no longer relevant. Thus we conclude [Theorem 1.1](#).

7. APPENDIX

7.1. Failure of Harnack Inequality. We include here some of the details of the counterexample of the Harnack inequality from [\[8\]](#).

Example 7.1 ([\[8, p.148\]](#)). For $k \in \mathbb{N}$ let I_k be any set of the form

$$I_k = (B_{4^{-k}}(\xi_k) \cup B_{4^{-k}}(-\xi_k)) \cap S^{n-1},$$

where $\xi_k \in \mathbb{S}^{n-1}$ are chosen such that the balls $B_{2^{-k}}(\xi_k)$ are mutually disjoint. Set

$$S = \{h \in \mathbb{R}^n \mid \frac{h}{|h|} \in \bigcup_{k \geq 1} I_k\},$$

and

$$K(h) = \mathbb{1}_S(h) |h|^{-n-\alpha}$$

for $|h| \neq 0$ and any fixed $\alpha \in (0, 2)$. Then it is shown in [8] that solutions u to $Lu = 0$ with L as in (1.1) do not need to satisfy a Harnack inequality. This set S allows to find a sequence of sets A_m and also a sequence of points x_m such that the probability of exiting B_1 and landing in A_m from x_m is significantly different than starting at 0. Specifically, if X is the stochastic process generated by this particular L , τ_{B_1} is the exit time from B_1 , and

$$u_m(x) = \mathbb{P}^x(X_{\tau_{B_1}} \in A_m),$$

then one obtains

$$Lu_m = 0 \text{ in } B_1$$

and

$$\frac{u_m(0)}{u_m(x_m)} \rightarrow \infty \text{ as } m \rightarrow \infty,$$

for an appropriately chosen sequence x_m . The main ideas behind the construction are similar to those of the counterexample for the case of singular measures presented in [4, Section 3].

In light of Example 7.1, it seems pertinent to point out where the proof of the Harnack inequality for \mathcal{A}_{CS} (see (7.3) below) from [9, Theorem 11.1] fails for \mathcal{A}_{sec} . A key step in [9, Theorem 11.1] is to find an equation satisfied by

$$w(x) = (cu(x_0) - u(x))^+,$$

for some appropriate c , given that $M_{\mathcal{A}_{CS}}^- u \leq 1$ in B_1 (plus some other properties). The details for the arguments we focus on appear in [9, p.629,630]. In the top half of p.630, a bound at one point, x_1 , is found on $M_{\mathcal{A}_{CS}}^- u(x_1)$, and then it is used via a shift in the integration variables to estimate $M_{\mathcal{A}_{CS}}^- w(x)$ at a different x . In order that the steps on p.630 would hold for the case of $K \in \mathcal{A}_{sec}$, one would basically need an estimate of the form

$$\frac{K(z - \hat{x})}{K(z)} \leq \tilde{c}r^{-n-\alpha}, \text{ for all } z \in \mathbb{R}^n \setminus B_r(\hat{z}),$$

where $\hat{x} \neq \hat{z}$ are certain special points. For $K \in \mathcal{A}_{sec}$, this will not hold in general because there is no way to rule out the possibility that $K(z) = 0$ when $K(z - \hat{x}) \neq 0$, and thus the argument in [9, p.630] will not carry over to the case of \mathcal{A}_{sec} . Given Example 7.1, it is fair to say this obstruction is permanent.

7.2. Pointwise evaluation for general F . We discuss in this section what one can expect from the regularity imposed on subsolutions to e.g. (1.3) by the definition of viscosity solutions at points where they can be touched from above by a test function. It is not true in general that the very convenient result, [9, Lemma 3.3], which gives

$$\int_{\mathbb{R}^n} |\delta_h^2 u(x)| |h|^{-n-\alpha} dh < \infty, \tag{7.1}$$

still holds when one considers classes of K more general than those of [9], such as \mathcal{A}_{sec} . The formula (7.1) seems to match with what is expected in the second order case, but is not a good guide for the integro-differential case. Indeed let \mathcal{M}^+ be the Pucci maximal operator of the

second order theory (see [12, Chapter 2]) and in the viscosity sense (for some bounded domain, Ω)

$$\mathcal{M}^+u \geq -f \text{ in } \Omega.$$

Then whenever $u - \phi$ has a global max at $x \in \Omega$, we have for the eigenvalues of $D^2u(x)$, $\{e_1, \dots, e_n\}$,

$$\lambda \sum_{e_i < 0} e_i + \Lambda \sum_{e_j > 0} e_j \geq -f(x)$$

which implies

$$f(x) + \Lambda \|\phi\|_{C^2} \geq \lambda \sum_{e_i < 0} (-e_i). \quad (7.2)$$

From this we deduce that u is also $C^{1,1}$ from below at x with a bound which depends on f , Λ , ϕ . However, as we show, even an analog of this to an integrated quantity such as (7.1) is too much to ask in the general integro-differential setting. The more generic analog to (7.2) is Corollary 7.7 below.

To make this precise, let

$$\mathcal{A}_{CS} = \left\{ K : \mathbb{R}^n \rightarrow \mathbb{R} : \frac{\lambda(2-\alpha)}{|h|^{-n-\alpha}} \leq K(h) \leq \frac{\Lambda(2-\alpha)}{|h|^{-n-\alpha}} \right\}. \quad (7.3)$$

Then without assuming u satisfies any equation, we have the more straightforward observation

Note 7.2. If $\phi \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$, $u - \phi$ has a global max at x , and $M_{\mathcal{A}_{CS}}^+ u(x) \geq 0$ classically, then

$$\int_{\mathbb{R}^n} (\delta_h^2 u(x))^- |h|^{-n-\alpha} dh \leq \frac{\Lambda}{\lambda} \int_{\mathbb{R}^n} (\delta_h^2 \phi(x))^+ |h|^{-n-\alpha} dh \leq \frac{C(n, \alpha) \Lambda}{\lambda} C(\phi).$$

We now present an example which illustrates the failure of Note 7.2 and puts in contrast $M_{\mathcal{A}_{CS}}^+$ to $M_{\mathcal{A}_{sec}}^+$.

Example 7.3. Let $n = 2$ and $\alpha > 1$. Define u as

$$u(x) = -|x| \phi \left(\theta \left(\frac{x}{|x|} \right) \right). \quad (7.4)$$

Here $\theta(\frac{x}{|x|}) \in [-\pi, \pi)$ is the angle of $\frac{x}{|x|}$, and ϕ is a smooth *even* angular cutoff such that

$$\phi(\theta) = \begin{cases} 1 & \text{if } |\theta| \in [0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi] \\ \text{smooth and monotone} & \text{if } |\theta| \in [\frac{\pi}{6}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{6}] \\ 0 & \text{if } |\theta| \in [\frac{\pi}{4}, \frac{3\pi}{4}]. \end{cases}$$

Let e_1, e_2 be the canonical basis vectors in \mathbb{R}^2 and V_1, V_2 be the sectors

$$V_1 = \left\{ z : \left| \left\langle \frac{z}{|z|}, e_1 \right\rangle \right| \geq \frac{\sqrt{3}}{2} \right\} \text{ and } V_2 = \left\{ z : \left| \left\langle \frac{z}{|z|}, e_2 \right\rangle \right| \geq \frac{\sqrt{3}}{2} \right\}.$$

We now list– but leave to the reader to check– some illustrative properties of u :

- $u - 0$ has a global max at $x = 0$
- $M_{\mathcal{A}_{sec}}^+ u(0) = 0$ by using $K(h) = \mathbb{1}_{V_2}(h) |h|^{-2-\alpha}$
- $M_{\mathcal{A}_{sec}}^- u(0) = -\infty$ by using $K(h) = \mathbb{1}_{V_1}(h) |h|^{-2-\alpha}$

$$\bullet \int_{\mathbb{R}^2} (\delta_h^2 u(x))^- |h|^{-2-\alpha} dh = +\infty.$$

We now state the pointwise evaluation results of [Section 3.3](#) for a general class of kernels. We hope these results can be useful elsewhere in the theory. Assume that \mathcal{A} is a class of kernels such that at least

$$K(h) \leq \Lambda |h|^{-n-\alpha} \text{ for all } h \in \mathbb{R}^n. \tag{7.5}$$

Then it is immediate that:

Note 7.4. If \mathcal{A} satisfies (7.5) then for each $R > 0$ $\left\{ \min(|h|^2, 1)K(h) : K \in \mathcal{A} \right\}$ is a uniformly integrable family of kernels on B_R .

Given [Note 7.4](#), the proof of [Proposition 3.1](#) (via [Lemma 3.2](#)) can be directly adapted to the case of $M_{\mathcal{A}}^+$. Furthermore, the pointwise evaluation also holds for any F which can be written as an inf – sup ([Proposition 7.6](#)).

Indeed, if K_m^* are optimizers (or achieve within ε of the supremum) for v_m (as in [Proposition 3.1](#)) then by [Note 7.4](#) there is a subsequence of $\min(|h|^2, 1)K_m^*(h)$ which converges pointwise a.e. x to some $K \in \mathcal{A}$. Then we can look at the pointwise convergence of

$$\frac{\delta_h^2 v_m(x)}{\min(|h|^2, 1)} \min(|h|^2, 1)K_m^*(h)$$

to conclude. This discussion proves [Proposition 7.5](#), and minor modifications yield [Proposition 7.6](#).

Proposition 7.5. *Assume that all $K \in \mathcal{A}$ satisfy (7.5). Then the assertion of [Proposition 3.1](#) remains true for $M_{\mathcal{A}_{sec}}^+$ replaced by $M_{\mathcal{A}}^+$.*

Proposition 7.6. *Assume that F is given by*

$$F(u, x) = \inf_{a \in \mathcal{S}} \sup_{K \in \mathcal{A}_a} \left\{ \int_{\mathbb{R}^n} \delta_h^2 u(x) K(h) dh \right\},$$

where \mathcal{S} is an arbitrary index set, and $\mathcal{A}_a \subset \mathcal{A}$ for all $a \in \mathcal{S}$. If u is a viscosity subsolution of $F(u, x) \geq f(x)$ and $u - \phi$ has a global maximum at x , then $F(u, x)$ is defined classically, and $F(u, x) \geq f(x)$.

A useful consequence of [Proposition 7.5](#) is:

Corollary 7.7. *If u is a viscosity subsolution of $F(u, x) \geq -f(x)$ in Ω and $u - \phi$ has a global maximum at $x \in \Omega$, then there exists at least one $K^* \in \mathcal{A}$ such that $(\delta_h^2 u)^-$ is integrable against K^* . Furthermore,*

$$\int_{\mathbb{R}^n} (\delta_h^2 u(x))^- K^*(h) dh \leq f(x) + \Lambda \sup_{K \in \mathcal{A}} \left(\int_{B_1} (\delta_h^2 \phi(x, y))^+ K(h) dh \right) + 1. \tag{7.6}$$

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