COMPLETE CURVATURE HOMOGENEOUS METRICS ON $\mathrm{SL}_2(\mathbb{R})$

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ABSTRACT. A construction is described that associates to each positive smooth function $F : S^1 \to \mathbb{R}$ a smooth Riemannian metric g_F on $\mathrm{SL}_2(\mathbb{R}) \cong \mathbb{R}^2 \times S^1$ that is complete and curvature homogeneous. The construction respects moduli: positive smooth functions F and G lie in the same $\mathrm{Diff}(S^1)$ orbit if and only if the associated metrics g_F and g_G lie in the same $\mathrm{Diff}(\mathrm{SL}_2(\mathbb{R}))$ orbit.

The constructed metrics all have curvature tensor modeled on the same algebraic curvature tensor. Moreover, the following are shown to be equivalent: F is constant, g_F is left-invariant, and $(\mathrm{SL}_2(\mathbb{R}), g_F)$ Riemannian covers a finite volume manifold. Applications of the construction are discussed.

1. Introduction

Let (M, g) be a connected Riemannian manifold, ∇ its Levi-Civita connection, and R its curvature tensor. Then (M, g) is said to be curvature homogeneous of order k if for every $p, q \in M$ there exists a linear isometry $I : T_pM \to T_qM$ such that

$$I^*(\nabla^i R)_q = (\nabla^i R)_p$$

for each $i = 0, 1, \ldots k$. When M is curvature homogeneous of order 0, M is simply said to be *curvature homogeneous*. Locally homogeneous (M, g) are clearly curvature homogeneous of all orders. In the seminal paper [12], I.M. Singer proved the converse:

Theorem 1.1 (Singer)**.** *A connected and complete d-dimensional Riemannian manifold (M, g) that is curvature homogeneous of order at least $d(d-1)/2 - 1$ is locally homogeneous. If, in addition, M is simply connected, then (M, g) is homogeneous.*

While Singer's theorem ensures that completeness and curvature homogeneity of sufficiently large order implies local homogeneity, there exist examples of complete and curvature homogeneous Riemannian manifolds that are not locally homogeneous. We refer the reader to [4] for an extensive collection of examples and additional references. In this note we prove:

Theorem 1.2. *There is a construction that associates to each positive smooth function $F : S^1 \to \mathbb{R}$ a complete and curvature homogeneous Riemannian metric g_F on $\mathrm{SL}_2(\mathbb{R})$. In this construction, the following are equivalent:*

1. *F is constant.*
2. *The metric g_F is left-invariant.*

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(3) $(\mathrm{SL}_2(\mathbb{R}), g_F)$ *Riemannian covers a finite volume manifold.*

Theorem 1.2 is related to a conjecture attributed to Gromov by Berger in [3] that we now describe. Let T denote a fixed algebraic curvature tensor on Euclidean space $\mathbb{E}^n$ and let M denote a connected, smooth n-manifold. A Riemannian metric h on M with curvature tensor R is said to be modeled on T if for each $x \in M$ there is a linear isometry $I : T_xM \to \mathbb{E}^n$ such that $I^*(T) = R_x$. It is clear that such a Riemannian metric h is curvature homogeneous and that $\mathrm{Diff}(M)$ acts on the space of such metrics by pullback. Let $\mathcal{M}(M, T)$ denote the space of $\mathrm{Diff}(M)$ orbits of *complete* Riemannian metrics on M with curvature tensor modeled on T.

Conjecture 1.1 (Gromov)**.** *If M is compact, then the moduli space $\mathcal{M}(M, T)$ is finite dimensional.*

It is known that the assumption of compactness in Gromov's conjecture cannot in general be replaced by an assumption of completeness on the metrics under consideration. For example, infinite dimensional moduli spaces of complete metrics with curvature tensors modeled on certain *reducible* symmetric spaces are constructed in [13] and [9] (see also [4][Propositions 4.15-4.16]). The following was posed in [13]:

Question (Tricerri and Vanhecke, Problem 2 in [13])**.** *Do the isometry classes of the germs of Riemannian metrics which have the curvature tensor of a given "irreducible" homogeneous Riemannian manifold depend on a finite number of parameters?*

As will be explained in Section 3, the Riemannian metrics constructed in Theorem 1.2 all have curvature tensors modeled on a fixed algebraic curvature tensor that we will call T throughout. The algebraic curvature tensor T is modeled on the curvature tensor of an irreducible left-invariant metric on $\mathrm{SL}_2(\mathbb{R})$. Our next theorem describes the moduli space of these metrics.

Theorem 1.3. *Let F and G be two positive smooth functions on the circle. Then there exists a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ such that $\Phi^*(g_G) = g_F$ if and only if there exists a diffeomorphism $\phi : S^1 \to S^1$ such that $F = \phi^*(G)$.*

The space of $\mathrm{Diff}(S^1)$ orbits of positive smooth functions on S^1 is easily seen to be infinite dimensional. Hence, Theorems 1.2 and 1.3 yield the following negative answer to Tricerri and Vanhecke's problem:

Corollary 1.4. *There is an algebraic curvature tensor T modeled on an irreducible left-invariant metric on $\mathrm{SL}_2(\mathbb{R})$ such that the moduli space $\mathcal{M}(\mathrm{SL}_2(\mathbb{R}), T)$ is infinite dimensional.*

Our construction also has an application to the problem of finding isocurved deformations of homogeneous Riemannian spaces. Let (M, g) be a homogeneous Riemannian manifold. In [8], Kowalski defines an *isocurved deformation* of g to be a family of smooth Riemannian metrics $\{g_t \,|\, t \in [0, 1]\}$ on M satisfying:

(1) Each (M, g_t) is a curvature homogeneous space with curvature tensor modeled on (M, g),
(2) The metrics g_t depend smoothly on t and $g_0 = g$,
(3) (M, g_{t_1}) is not locally isometric to (M, g_{t_2}) when $t_1 \neq t_2$.

If in addition, the metrics g_t with $t \in (0, 1)$ are not locally homogeneous, then the isocurved deformation is said to be *proper*.

A proper isocurved defomation of an irreducucible homogeneous metric g_0 on the three-dimenisonal Lie group $E(1,1)$ is constructed in [8]. However, the metric g_1 in the deformation is not complete, and the completeness of the intermediate metrics is not determined. Problem 1 in [8] asks to find a proper isocurved deformation of an irreducible homogenous Riemannian manifold through *complete* Riemannian metrics.

Corollary 1.5. *Let $F : S^1 \to \mathbb{R}$ be a non-constant smooth positive function and let $F_t = (1-t) + tF$. Then the family of metrics*

$$\{g_t = g_{F_t} \,|\, t \in [0,1]\}$$

is a proper isocurved deformation of the irreducible homogeneous Riemannian manifold $(\mathrm{SL}_2(\mathbb{R}), g_1)$ through complete Riemannian metrics.

Proof. As remarked above, each of the metrics g_t is modeled on a fixed algebraic curvature tensor T; their smoothness in the parameter t will be evident from the construction. The metric g_0 is homogeneous, each of the metrics g_t is complete, and each of the metrics g_t with $t > 0$ is not locally homogeneous by Theorem 1.2; the irreducibility of the metric g_0 is clear. It remains to check that the metrics g_t are pairwise non-isometric. This follows from Theorem 1.3 after checking that the functions F_t pairwise lie in different $\mathrm{Diff}(S^1)$ orbits. This is an immediate consequence of the fact that the number of critical points and the associated critical values of smooth functions on S^1 are $\mathrm{Diff}(S^1)$-invariants. □

Theorem 1.2 is also related to a classification result for *constant vector curvature* three-manifolds contained in [10] that will be used in Section 3. A Riemannian manifold (M, g) has constant vector curvature ε if each tangent vector $v \in TM$ lies in a tangent plane of sectional curvature ε. This curvature condition was introduced as a pointwise analogue of the higher rank condition for Riemannian manifolds. Motivated by a number of results on rank-rigidity such as [1, 2, 5, 6, 7, 11] the present authors proved the following rigidity result for constant vector curvature -1 three-manifolds [10, Theorem 1.1]:

Theorem 1.6. *Suppose that M is a finite volume three-manifold with constant vector curvature -1. If $\sec \leq -1$, then M is real hyperbolic. If $\sec \geq -1$ and M is not real hyperbolic, then its universal covering is isometric to a left-invariant metric on one of the Lie Groups $E(1,1)$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with sectional curvatures having range $[-1,1]$.*

As will be explained in Section 3, the metrics constructed in Theorem 1.2 all have constant vector curvature -1 and sectional curvatures having range $[-1,1]$. Therefore, it is not possible to remove the finite volume hypothesis in Theorem 1.6 in the case when $\sec \geq -1$.

2. $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R})$ denote the Lie group consisting of 2×2 real matrices of determinant one and let $e \in \mathrm{SL}_2(\mathbb{R})$ denote the identity element. Its Lie algebra $\mathrm{sl}_2(\mathbb{R}) \cong T_e\,\mathrm{SL}_2(\mathbb{R})$ consists of 2×2 real matrices with trace equal to zero. Consider the following three one-parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$:

$$K = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \,|\, \theta \in \mathbb{R} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The multiplication map $K \times N \times A \to \mathrm{SL}_2(\mathbb{R})$, $(k, n, a) \mapsto kna$ is a diffeomorphism, yielding the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = KNA$.

Define trace zero matrices $E_1, E_2, E_3 \in \mathrm{sl}_2(\mathbb{R})$ by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } E_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Then $\{E_1, E_2, E_3\}$ is a basis for the Lie algebra $\mathrm{sl}_2(\mathbb{R})$. Moreover, E_1, E_2, and E_3 are the infinitesimal generators of the one-parameter subgroups K, N, and A, respectively. This Lie algebra basis satisfies the following bracket relations:

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = E_1 - 2E_2. \tag{2.1}$$

The vectors E_i have unique extensions to left-invariant vector fields on $\mathrm{SL}_2(\mathbb{R})$ that we also denote by E_i. Declaring the left-invariant framing $\{E_1, E_2, E_3\}$ of $\mathrm{SL}_2(\mathbb{R})$ to be orthonormal determines a left-invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$. Throughout the remainder of this paper, we let g_1 denote this left-invariant metric. The pull back of its curvature tensor via a linear isometry from Euclidean space $\mathbb{E}^3$ to $T_e \mathrm{SL}_2(\mathbb{R})$ defines an algebraic curvature tensor that we denote by T in the remainder of the paper. In the next section, we give the construction of Theorem 1.2. The metrics constructed will all have curvature tensors modeled on the algebraic curvature tensor T.

3. The Construction

Note that the subgroup K of $\mathrm{SL}_2(\mathbb{R})$ is diffeomorphic to S^1. Throughout what follows, we assume that a diffeomorphism between K and S^1 has been fixed, identifying positive smooth functions on K with those on S^1. A positive smooth function $F : K \to \mathbb{R}$ determines a positive smooth function $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ as follows. Given $g \in \mathrm{SL}_2(\mathbb{R})$, there is a unique expression $g = kna$ with $k \in K$, $n \in N$, and $a \in A$ by the Iwasawa decomposition. Define $\bar{F}(g) = \bar{F}(kna) = F(k)$.

Alternatively, the bracket relations (2.1) show that the left-invariant vector fields E_2 and E_3 span an involutive plane distribution; the foliation of $\mathrm{SL}_2(\mathbb{R})$ by integral surfaces of this distribution coincides with the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of the subgroup NA. As NA is a closed subgroup of $\mathrm{SL}_2(\mathbb{R})$, the natural projection map to the space of left-cosets

$$\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) / NA$$

is smooth. Note that the space of cosets $\mathrm{SL}_2(\mathbb{R}) / NA$ is diffeomorphic to K. Then $\bar{F} = F \circ \pi$ is constant on the leaves of the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of NA. We summarize this in the following lemma.

Lemma 3.1. *Smooth functions $F : K \to \mathbb{R}$ lift to smooth functions $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ satisfying $E_2(\bar{F}) = E_3(\bar{F}) = 0$.*

Let $F : K \to \mathbb{R}$ be a smooth and positive function and $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ its associated lift. Define a framing $\{e_1, e_2, e_3\}$ of $\mathrm{SL}_2(\mathbb{R})$ by

$$e_1 = \bar{F}E_1, \quad e_2 = E_2, \quad e_3 = E_3. \tag{3.1}$$

We will call such a framing an F-framing. The bracket relations for an F-framing are easy to deduce from (2.1) and the fact that $E_2(\bar{F}) = E_3(\bar{F}) = 0$. They are given by

$$[e_1, e_2] = 2\bar{F}e_3, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = e_1 - 2\bar{F}e_2. \tag{3.2}$$

Definition 3.1. Given a smooth positive function $F : K \to \mathbb{R}$, the F-metric on $\mathrm{SL}_2(\mathbb{R})$ is the Riemannian metric denoted by g_F which is defined by declaring the associated F-framing to be g_F orthonormal.

Note that for the function F which is identically one on K, the associated F-metric is the left-invariant metric g_1 described in Section 2. We remark that the space of F-metrics is path connected. Indeed, given two positive functions F_0 and F_1 on K, the metrics $g_{(1-t)F_0+tF_1}$ with $t \in [0, 1]$ defines the path joining g_{F_0} to g_{F_1}. As we shall show, all F-metrics have curvature tensors modeled on the algebraic curvature tensor T.

In order to calculate the curvatures of an F-metric, we first calculate the Christoffel symbols. As an F-framing is by definition orthonormal for the metric g_F, Koszul's formula reads:

$$g_F(\nabla_{e_i} e_j, e_k) = \frac{1}{2}\{g_F([e_i, e_j], e_k) - g_F([e_j, e_k], e_i) + g_F([e_k, e_i], e_j)\}. \tag{3.3}$$

Combining (3.2) and (3.3), yields:

$$\begin{aligned} \nabla_{e_1} e_3 &= e_1 - 2\bar{F}\, e_2 & \nabla_{e_2} e_3 &= -e_2 \\ \nabla_{e_3} e_1 &= 0 & \nabla_{e_3} e_2 &= 0 \\ \nabla_{e_2} e_1 &= 0 & \nabla_{e_2} e_2 &= e_3 \\ \nabla_{e_1} e_2 &= 2\bar{F}\, e_3 & \nabla_{e_1} e_1 &= -e_3 \\ \nabla_{e_3} e_3 &= 0. \end{aligned} \tag{3.4}$$

We let R_{ijkl} denote the component of the curvature tensor

$$R(e_i, e_j, e_k, e_l) = g_F(\nabla_{e_i}\nabla_{e_j} e_k - \nabla_{e_j}\nabla_{e_i} e_k - \nabla_{[e_i,e_j]} e_k, e_l).$$

Tedious but straightforward calculations using (3.2), (3.4), and the fact that $e_2(\bar{F}) = e_3(\bar{F}) = 0$ show that:

$$R_{1221} = 1, \quad R_{1331} = -1 = R_{2332}, \quad R_{ijkl} = 0 \ \text{ if three indices are distinct.} \tag{3.5}$$

The symmetries of the curvature tensor determine its remaining components.

Corollary 3.2. *An F-metric g_F is curvature homogeneous and has curvature tensor modeled on the algebraic curvature tensor T. An F-framing diagonalizes the*

Ricci tensor. If σ is a two plane and $v = \sum_{i=1}^{3} c_i e_i$ is a unit vector orthogonal to σ, then

$$\sec(\sigma) = c_3^2 - c_1^2 - c_2^2.$$

Consequently, g_F has constant vector curvature -1, e_3 lies in the intersection of all curvature -1 planes and the range of sectional curvatures for an F-metric is $[-1, 1]$.

Proof. To prove the first claim, note that by (3.5), the curvatures of an F-metric with respect to an F-framing do not depend on the function $F : K \to \mathbb{R}$. Therefore, they all have curvature tensors modeled on the curvature tensor of the F-metric corresponding to $F \equiv 1$ which is the left-invariant metric g_1 constructed at the end of the previous section.

The fact that an F-framing diagonalizes the Ricci tensor is immediate from (3.5). This fact and [10, Lemma 2.2] yield the curvature formula. The curvature formula implies the last statement. □

Lemma 3.3. *An F-metric g_F is complete.*

Proof. Let $F : K \to \mathbb{R}$ be a positive smooth function and g_F the associated F-metric. As K is compact, there exists $M > 1$ such that $\frac{1}{M} < F < M$. Consider the Riemannian metrics $M^{-2}g_1$ and $M^2 g_1$ obtained by scaling the left-invariant metric g_1. The induced norms satisfy

$$M^{-1}\|v\|_{g_1} = \|v\|_{M^{-2}g_1} < \|v\|_{g_F} < \|v\|_{M^2 g_1} = M\|v\|_{g_1}$$

for each tangent vector $v \in T\,\mathrm{SL}_2(\mathbb{R})$. Consequently, the induced path metrics satisfy

$$M^{-1} d_{g_1}(p, q) \le d_{g_F}(p, q) \le M d_{g_1}(p, q)$$

for any pair of points $p, q \in \mathrm{SL}_2(\mathbb{R})$. As d_{g_1} Cauchy sequences converge, the same is true of d_{g_F} Cauchy sequences. □

The following lemma may be of interest to some readers. It is not used in the proof of our main results and may be skipped.

Lemma 3.4. *For any F-metric g_F, the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of NA is a foliation by totally geodesic hyperbolic planes.*

Proof. Let $F : K \to \mathbb{R}$ be a smooth positive function, g_F the associated F-metric, and $\{e_1, e_2, e_3\}$ the associated F-framing. The leaves of the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left cosets of NA are precisely the integral surfaces of the involutive plane distribution $e_2 \wedge e_3$. These leaves are totally geodesic since by (3.4), $\nabla_{e_2} e_1 = \nabla_{e_3} e_1 = 0$. By (3.5), $R_{2332} = -1$ so that the leaves are hyperbolic. As NA is diffeomorphic to $\mathbb{R}^2$, the leaves are hyperbolic planes. □

To complete the proof of Theorem 1.2 from the introduction, it remains to establish the following proposition.

Proposition 3.5. *For a positive smooth function $F : K \to \mathbb{R}$, the following are equivalent:*

(1) *F is constant.*
(2) *The metric g_F is left-invariant.*
(3) $(\mathrm{SL}_2(\mathbb{R}), g_F)$ *Riemannian covers a finite volume manifold.*

Proof. Let $F : K \to \mathbb{R}$ be a positive smooth function and g_F the associated F-metric on $\mathrm{SL}_2(\mathbb{R})$.

Proof that (1) $\implies$ (2):
As F is constant, so is its lift $\bar{F}$. The associated F-framing $\{e_1 = \bar{F}E_1, e_2 = E_2, e_3 = E_3\}$ is easily seen to be left-invariant since the framing $\{E_1, E_2, E_3\}$ is left-invariant. Therefore g_F is a left-invariant metric.

Proof that (2) $\implies$ (3):
This is an easy consequence of the fact that $\mathrm{SL}_2(\mathbb{R})$ admits lattice subgroups.

Proof that (3) $\implies$ (1):
Let M denote the finite volume manifold Riemannian covered by $(\mathrm{SL}_2(\mathbb{R}), g_F)$. We first claim that the metric g_F is locally homogeneous. Indeed, by Corollary 3.2, M has constant vector curvature -1 and sectional curvatures with range $[-1, 1]$. By Theorem 1.6, the universal covering $(\widetilde{\mathrm{SL}_2(\mathbb{R})}, \tilde{g}_F)$ is left-invariant (and homogeneous), whence g_F is locally homogeneous.

Let $\bar{F}$ denote the lift of F to $\mathrm{SL}_2(\mathbb{R})$ and let $\{e_1, e_2, e_3\}$ be the associated F-framing. Let $p, q \in \mathrm{SL}_2(\mathbb{R})$ be two points. As g_F is locally homogeneous, there is an $r > 0$ and an isometry I between the balls of radius r centered at p and q with $I(p) = q$:

$$I : B(p, r) \to B(q, r).$$

The derivative map $dI : TB(p, r) \to TB(q, r)$ preserves the line field spanned by e_3 and the perpendicular plane field $e_1 \wedge e_2$ by the curvature formula in Corollary 3.2. Therefore, there exists a smooth map $\theta : B(q, r) \to \mathbb{R}$ such that $dI(e_3) = \pm e_3$ and such that the restriction of dI to the plane field $e_1 \wedge e_2$ has matrix representation given by either $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ or $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ with respect to the $\{e_1, e_2\}$ framing.

By (3.2),

$$dI_p([e_1, e_2]_p) = dI_p(2\bar{F}(p)e_3) = \pm 2\bar{F}(p)e_3 \in T_q\,\mathrm{SL}_2(\mathbb{R})$$

where the sign is $+$ if dI preserves the orientation of e_3 and is $-$ if the orientation is reversed. A simple calculation yields,

$$g_F([dI_p(e_1), dI_p(e_2)]_q, e_3)_q = \pm[e_1, e_2]_q = \pm 2\bar{F}(q)$$

where the sign is $+$ if dI preserves the orientation of the plane field $e_1 \wedge e_2$ and is $-$ if the orientation is reversed.

Since, $dI_p([e_1, e_2]_p) = [dI_p(e_1), dI_p(e_2)]_q$, we have that $\bar{F}(p) = \pm\bar{F}(q)$. As $\bar{F}$ is everywhere positive, it must be the case that $\bar{F}(p) = \bar{F}(q)$. Therefore F is constant, concluding the proof. □

We conclude the paper with a proof of Theorem 1.3, restated for the reader's convenience, followed by a conjecture.

Theorem 3.6. *Let F and G be two positive smooth functions on the circle. Then there exists a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ such that $\Phi^*(g_G) = g_F$ if and only if there exists a diffeomorphism $\phi : S^1 \to S^1$ such that $F = \phi^*(G)$.*

Proof. Recall that a diffeomorphism between S^1 and K has been fixed, identifying positive smooth functions on these two spaces.

First, assume that there is a diffeomorphism $\phi : K \to K$ such that $\phi^*(G) = F$. Define a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ as follows. By the Iwasawa decomposition each $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique expression $g = kna$; define $\Phi(g) = \Phi(kna) = \phi(k)na$. It is routine to check that $\Phi^*(g_G) = g_F$.

Now assume that $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ is a diffeomorphism satisfying $\Phi^*(g_G) = g_F$. Let $\bar{F}$ and $\bar{G}$ denote the lifts of F and G to $\mathrm{SL}_2(\mathbb{R})$ and let $\{e_1, e_2, e_3\}$ and $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ denote the associated F-framing and G-framing of $T\,\mathrm{SL}_2(\mathbb{R})$, respectively. Since $e_2 = \tilde{e}_2$, $e_3 = \tilde{e}_3$, and e_1 and $\tilde{e}_1$ are positively parallel, these framings induce the same orientation of $\mathrm{SL}_2(\mathbb{R})$.

As $\Phi : (\mathrm{SL}_2(\mathbb{R}), g_F) \to (\mathrm{SL}_2(\mathbb{R}), g_G)$ is an isometry, it preserves the sectional curvatures of planes. By Corollary 3.2 it follows that the derivative map

$$d\Phi : T\,\mathrm{SL}_2(\mathbb{R}) \to T\,\mathrm{SL}_2(\mathbb{R})$$

satisfies $d\Phi(e_3) = \pm\tilde{e}_3$ and maps the plane field $e_1 \wedge e_2$ isometrically to the plane field $\tilde{e}_1 \wedge \tilde{e}_2$. Therefore, there exists a smooth map

$$\theta : (\mathrm{SL}_2(\mathbb{R}), g_G) \to \mathbb{R}$$

such that the matrix representation of

$$d\Phi|_{e_1\wedge e_2} : e_1 \wedge e_2 \to \tilde{e}_1 \wedge \tilde{e}_2$$

with respect to the ordered framings $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ is given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

if $d\Phi|_{e_1\wedge e_2}$ preserves orientation or by

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

if $d\Phi|_{e_1\wedge e_2}$ reverses orientation.

By (3.2),

$$d\Phi([e_1, e_2]) = d\Phi(2\bar{F}e_3) = \pm 2\bar{F}\tilde{e}_3.$$

A simple calculation shows that

$$[d\Phi(e_1), d\Phi(e_2)] = \pm\left(-\tilde{e}_1(\theta)\,\tilde{e}_1 - \tilde{e}_2(\theta)\,\tilde{e}_2 + 2\bar{G}\,\tilde{e}_3\right)$$

where the sign $\pm$ is $+$ if and only if $d\Phi|_{e_1\wedge e_2}$ is orientation preserving.

Since $d\Phi([e_1, e_2]) = [d\Phi(e_1), d\Phi(e_2)]$, comparing $\tilde{e}_3$ components, we have that $\bar{F} = \pm\Phi^*(\bar{G})$. As both $\bar{F}$ and $\bar{G}$ are positive, we have

$$\bar{F} = \Phi^*(\bar{G}). \tag{3.6}$$

Consequently, $d\Phi(e_3) = \tilde{e}_3$ if and only if $d\Phi|_{e_1\wedge e_2}$ is orientation preserving. In particular, Φ is orientation preserving.

Comparing $\tilde{e}_1$ and $\tilde{e}_2$ components yields

$$\tilde{e}_1(\theta) = \tilde{e}_2(\theta) = 0. \tag{3.7}$$

By (3.2) and (3.7),

$$2\bar{G}\tilde{e}_3(\theta) = [\tilde{e}_1, \tilde{e}_2](\theta) = (\tilde{e}_1\tilde{e}_2 - \tilde{e}_2\tilde{e}_1)(\theta) = 0.$$

As $\bar{G}$ is nonzero, it follows that $\tilde{e}_3(\theta) = 0$, whence $\theta : (\mathrm{SL}_2(\mathbb{R}), g_G) \to \mathbb{R}$ is globally constant. In what follows, we will consider the two cases $d\Phi(e_3) = \tilde{e}_3$ and $d\Phi(e_3) = -\tilde{e}_3$ separately.

Case I: *The case when* $d\Phi(e_3) = \tilde{e}_3$.

As Φ is orientation preserving, we have that $d\Phi|_{e_1 \wedge e_2}$ is orientation preserving. Using (3.2),

$$g_G(d\Phi([e_2, e_3]), \tilde{e}_1) = \sin(\theta).$$

Using (3.2),

$$g_G([d\Phi(e_2), d\Phi(e_3)], \tilde{e}_1) = -\sin(\theta).$$

As $d\Phi([e_2, e_3]) = [d\Phi(e_2), d\Phi(e_3)]$, it follows that $\sin(\theta) = 0$ and that θ is an integral multiple of π.

As θ is an integral multiple of π, the derivative map $d\Phi$ preserves the plane distribution $e_2 \wedge e_3$. Consequently, the diffeomorphism Φ preserves the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of NA and descends to a diffeomorphism ϕ of K. By (3.6), $F = \phi^*(G)$, concluding the proof in this case.

Case II: *The case when* $d\Phi(e_3) = -\tilde{e}_3$.

As Φ is orientation preserving, we have that $d\Phi|_{e_1 \wedge e_2}$ is orientation reversing. Using (3.2),

$$g_G(d\Phi([e_2, e_3]), \tilde{e}_2) = \cos(\theta).$$

Using (3.2),

$$g_G([d\Phi(e_2), d\Phi(e_3)], \tilde{e}_2) = 2\bar{G}\sin(\theta) - \cos(\theta).$$

As $d\Phi([e_2, e_3]) = [d\Phi(e_2), d\Phi(e_3)]$, it follows that $\cos(\theta) = \bar{G}\sin(\theta)$. As θ is constant, so is $\bar{G}$. By (3.6) $\bar{F} = \bar{G}$ are equal constants. Hence, $F = G$ are equal constants, concluding the proof. □

We conclude with the following:

Conjecture 3.1. *The metrics g_F constructed in this paper describe all of the complete Riemannian metrics on* $\mathrm{SL}_2(\mathbb{R})$ *(up to isometry) that are modeled on the curvature tensor T.*

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