# REAL PROJECTIVE SPACES WITH ALL GEODESICS CLOSED

Abstract. Let $M$ be a smooth $n$-manifold admitting an even degree smooth covering by a smooth homotopy $n$-sphere. When $n \neq 3$, we prove that if $g$ is a Riemannian metric on $M$ with all geodesics closed, then $g$ has constant sectional curvatures. In particular, $n$-dimensional real projective spaces ($n \neq 3$) with all geodesics closed have constant sectional curvatures.

## 1. Introduction

A Riemannian metric on a compact smooth manifold is *Blaschke* when its injectivity radius equals its diameter, and following [17], is *Besse* when all of its geodesics are closed. We prove the following:

Theorem 1. A Besse metric $g$ on a smooth homotopy sphere is Blaschke if

(1) all prime closed geodesics have equal length, and
(2) $g$ admits a fixed point free isometric involution.

The standard smooth sphere of each dimension admits Besse metrics with non-constant sectional curvatures [1, Chapter 4]. Contrastingly, if a smooth homotopy sphere admits a Blaschke metric, then the sphere is smoothly standard and the metric has constant sectional curvatures [1, Appendix D]. Conjecturally, (1) holds for Besse metrics on simply-connected manifolds, and is known to hold for Besse metrics on smooth homotopy spheres of dimension other than three [8, 15, 17]. Combining these results with Theorem 1 immediately implies the following:

Theorem 2. Let $M$ be a smooth $n$-manifold admitting an even degree smooth covering by a smooth homotopy $n$-sphere. When $n \neq 3$, if $g$ is a Besse metric on $M$, then $g$ has constant sectional curvatures.

As a corollary, topologically or smoothly exotic real projective spaces as in [2, 4, 5, 7, 10, 11] do not admit Besse metrics. Previously known results concern Besse metrics on smoothly standard real projective spaces. In this case, the constant curvature metrics were known to be infinitesimally rigid and to not admit nontrivial conformal Besse deformations [6, 12, 13]. Theorem 2 was established for surfaces via clever dynamical arguments [16]; the geometric approach herein specializes to an alternative proof for surfaces.

## 2. Proof of Theorem 1

Let $X$ denote a Riemannian manifold with the Besse property whose underlying smooth manifold is an $n$-dimensional homotopy sphere. Further assume that $X$

satisfies the hypotheses of Theorem 1. Rescale $X$, if necessary, so that prime closed geodesics have length 2 and let $A : X \to X$ be a fixed point free isometric involution.

**Lemma 2.1.** ***(a)*** *The diameter of $X$ equals* 1.
***(b)*** *For each $x \in X$, $d(x, A(x)) = 1$.*
***(c)*** *The injectivity and conjugate radii of $X$ are equal.*

*Proof. (a).* As prime closed geodesics have length 2, no geodesic segment of length greater than 1 is minimizing. Therefore, if some minimizing geodesic segment has length 1, then $X$ has diameter 1.

Choose $p \in X$ minimizing the positive continuous function $x \mapsto d(x, A(x))$ and a minimizing geodesic segment $\gamma$ joining $p$ to $A(p)$. We claim that $\gamma$ has length 1. To establish this claim, it suffices to prove that $\gamma$ and $A(\gamma)$ meet smoothly at $p$ and $A(p)$ to form a closed geodesic of length 2.

Let $m$ denote the midpoint of $\gamma$. If $\gamma$ and $A(\gamma)$ fail to meet smoothly at $p$ or at $A(p)$, then $\gamma \cup A(\gamma)$ contains a piecewise geodesic path with one non-smooth point (at $p$ or $A(p)$) joining $m$ to $A(m)$ of length equal to that of $\gamma$. In this case, the strict triangle inequality implies $d(m, A(m)) < d(p, A(p))$ [1, 5.15], contrary to the choice of $p$.

*(b).* For $x \in X$, $1 = d(p, A(p)) \leqslant d(x, A(x)) \leqslant 1$ by the proof of (a).

*(c).* The injectivity radius of $X$ equals the minimum of the conjugate radius of $X$ and half the length of the shortest closed geodesic in $X$. By (a), the latter equals the diameter. Consequently, either (c) holds or $X$ is Blaschke. This concludes the proof since (c) is known to hold in a simply connected Blaschke manifold [1, Chapter 5, Section C]. ☐

Given $x, y \in X$, let $\Lambda(x, y)$ denote the path space consisting of piecewise smooth maps $c : [0, 1] \to X$ with $c(0) = x$ and $c(1) = y$. Let $E : \Lambda(x, y) \to \mathbb{R}$ denote the energy function. For $e > 0$, set

$$\Lambda_e(x, y) = E^{-1}([0, e))$$

and

$$\Lambda^e(x, y) = E^{-1}([0, e]).$$

Let $L : \Lambda(x, y) \to \mathbb{R}$ denote the length function. The inequality $L^2(c) \leqslant E(c)$ holds for each $c \in \Lambda(x, y)$ with equality if and only if $c$ is parameterized proportionally to arc-length. Critical points of $E$ consist of geodesic segments joining $x$ to $y$ parameterized proportionally to arc-length. In particular, $L^2(\gamma) = E(\gamma)$ for critical points $\gamma$ of $E$.

**Lemma 2.2.** *The critical values of $E : \Lambda(x, A(x)) \to \mathbb{R}$ are $\{(2k+1)^2 \mid k \in \mathbb{N} \cup \{0\}\}$ for each $x \in X$.*

*Proof.* By Lemma 2.1, $x$ and $A(x)$ are distance 1 apart in a closed geodesic of length 2. A geodesic segment starting at $x$, wrapping around this closed geodesic $k$ times, and then ending at $A(x)$ has length $2k + 1$. Thus, $(2k + 1)^2$ is a critical value for each $k \in \mathbb{N} \cup \{0\}$.

To show that there are no additional critical values, let $\gamma : \mathbb{R} \to X$ be a unit-speed geodesic with $\gamma(0) = x$ and $\gamma(l) = A(x)$ for some $l > 0$. We must show that $l = 2k + 1$ for some $k \in \mathbb{N} \cup \{0\}$.

As the map $\gamma$ is 2-periodic, there exists a unique $k_0 \in \mathbb{N} \cup \{0\}$ such that $2k_0 < l < 2k_0 + 2$. If $l < 2k_0 + 1$, then the restriction of $\gamma$ to the interval $[2k_0, l]$ is a geodesic segment joining $x$ to $A(x)$ of length less than 1, a contradiction. If $l > 2k_0 + 1$ then the restriction of $\gamma$ to the interval $[l, 2k_0 + 2]$ is a geodesic segment joining $A(x)$ to $x$ of length less than 1, a contradiction. Therefore, $l = 2k_0 + 1$. □

By the Morse Index Theorem, the index of a critical point $\gamma \in \Lambda(x, y)$ equals the number of parameters $s \in (0, 1)$ for which $\gamma(s)$ is conjugate to $\gamma(0)$ along $\gamma$, counted with multiplicities.

Let $UX$ denote the unit-sphere bundle of $X$ and for $u \in UX$, let

$$\gamma_u : [0, \infty) \to X$$

denote the geodesic map determined by $\dot{\gamma}_u(0) = u$. The location of conjugate points along a family of geodesics (when counted with multiplicities) depends continuously on the geodesics' initial velocity vectors. This well-known fact is the content of the next lemma [1, 1.98 Proposition].

**Lemma 2.3.** *Let $u_0 \in UX$. If the $k$th conjugate point to $\gamma_{u_0}(0)$ along the geodesic $\gamma_{u_0}$ occurs at time $s_0$, then there exists an interval $[s', s'']$ with $s' < s_0 < s''$ and a neighborhood $\mathcal{U}$ of $u_0$ in $UX$ such that, for $u \in \mathcal{U}$, the $k$th conjugate point of $\gamma_u(0)$ along the geodesic $\gamma_u$ occurs at a time $s$ satisfying $s' < s < s''$.*

**Lemma 2.4.** *Geodesic segments of length greater than 2 have index at least $n$.*

*Proof.* Define $G \subset UX$ by $u \in G$ if and only if the first conjugate point to $\gamma_u(0)$ along $\gamma_u(s)$ occurs before time 2. As all prime closed geodesics have length 2, for each $v \in UX$, the point $\gamma_v(2)$ is conjugate to $\gamma_v(0)$ along $\gamma_v(s)$ with multiplicity $n - 1$. Thus, for each $u \in G$, the $n$th conjugate point to $\gamma_u(0)$ along $\gamma_u(s)$ occurs no later than time 2. Consequently, it suffices to prove that $G = UX$.

The set $G$ is nonempty and open by Lemmas 2.1 and 2.3. To prove that $G$ is closed let $\{u_i\} \subset G$ be a sequence converging to $v \in UX$. Lemma 2.3 implies that the $n$th conjugate point to $\gamma_v(0)$ along $\gamma_v(s)$ occurs no later than time 2. As multiplicities of conjugate points cannot exceed $n - 1$, the first conjugate point occurs before time 2 so that $v \in G$. □

Given $x, y \in X$ and a continuous map $h : S^{n-1} \to \Lambda(x, y)$, define the *associated map*

$$\hat{h} : S^{n-1} \times [0, 1] \to X$$

by $\hat{h}(\theta, t) = h(\theta)(t)$ for each $(\theta, t) \in S^{n-1} \times [0, 1]$.

If the map $h$ represents a nontrivial class in $\pi_{n-1}(\Lambda(x, y)) \cong \mathbb{Z}$, then the associated map $\hat{h}$ is surjective.

**Lemma 2.5.** *For $x \in X$, if there exists a continuous map $g : S^{n-1} \to \Lambda_9(x, A(x))$ with nontrivial homotopy class, then the injectivity radius at $x$ equals 1.*

*Proof.* Fix a positive $\epsilon$ smaller than the injectivity radius at $A(x)$ and let $S$ denote the $\epsilon$-sphere with center $A(x)$. By Lemma 2.1 and the strict triangle inequality, for each $y \in S$, $d(x,y) \geqslant 1-\epsilon$ with equality only if $y$ lies along a minimizing geodesic of length 1 joining $x$ to $A(x)$. Therefore, to prove the Lemma, it suffices to prove that $d(x,y) = 1-\epsilon$ for each $y \in S$. We argue by contradiction assuming that for some $y_0 \in S$, $d(x,y_0) > 1-\epsilon$. Set $d_0 = d(x,y_0)$ and note that by the strict triangle inequality $d_0 < 1+\epsilon$. In particular,

$$1 < (d_0+\epsilon)^2 < (1+2\epsilon)^2 < 9.$$

By Lemma 2.2, the only critical value of $E : \Lambda(x, A(x)) \to \mathbb{R}$ less than 9 is 1. Standard Morse theoretic arguments, e.g. [14, Theorems 3.1 and 16.2], imply that $g$ is homotopic to a map $h$ having image in $\Lambda_{(d_0+\epsilon)^2}(x, A(x))$.

As $\hat{h}$ is surjective, there exists a $(\theta_0, t_0) \in S^{n-1} \times [0,1]$ such that $y_0 = h(\theta_0)(t_0)$. As the length of $h(\theta_0)$ is less than $d_0+\epsilon$ and the length of its restriction to the interval $[t_0, 1]$ is at least $\epsilon$, its restriction to the interval $[0, t_0]$ has length less than $d_0$, a contradiction.

□

Given $x, y \in X$ with $d(y, A(x)) = \epsilon < 1$, let $\tau : [0,\epsilon] \to X$ be a unit-speed geodesic with $\tau(0) = y$ and $\tau(\epsilon) = A(x)$. Given $c \in \Lambda(x,y)$ define $\tau * c \in \Lambda(x, A(x))$ by

$$\tau * c(t) = \begin{cases} c(\frac{t}{1-\epsilon}) & \text{for } t \in [0, 1-\epsilon] \\ \tau(t-(1-\epsilon)) & \text{for } t \in [1-\epsilon, 1]. \end{cases}$$

The energies of $c$ and $\tau * c$ satisfy $E(\tau * c) = \frac{E(c)}{1-\epsilon} + \epsilon$.

Given a map $f : S^{n-1} \to \Lambda(x,y)$, define a map $\tau f : S^{n-1} \to \Lambda(x, A(x))$ by $\tau f(\theta) = \tau * f(\theta)$ for each $\theta \in S^{n-1}$. Note that $\tau f$ has nontrivial homotopy class in $\pi_{n-1}(\Lambda(x, A(x)))$ if and only if $f$ has nontrivial homotopy class in $\pi_{n-1}(\Lambda(x,y))$.

**Proof of Theorem 1.** Choose $x \in X$ realizing the injectivity radius of $X$. By Lemma 2.5, it suffices to construct a map $g : S^{n-1} \to \Lambda_9(x, A(x))$ with nontrivial homotopy class. The map $g$ that we construct has the form $g = \tau f$ as above for an appropriate choice of $\tau$ and $f$ that we now describe.

The geodesic segment $\tau$ is chosen as follows. Choose a point $y \in X$ such that $y$ is not conjugate to $x$ along any geodesic in $X$ and such that $d(y, A(x))$ is less than the minimum of 1/2 and the injectivity radius of $X$. Set $\epsilon = d(y, A(x))$. Let $\tau : [0,\epsilon] \to X$ be the unit-speed minimizing geodesic determined by $\tau(0) = y$ and $\tau(\epsilon) = A(x)$.

The map $f : S^{n-1} \to \Lambda(x,y)$ is constructed as follows. Choose a continuous map $\bar{f} : S^{n-1} \to \Lambda(x,y)$ with a nontrivial homotopy class. The choice of the point $y$ ensures that critical points of $E : \Lambda(x,y) \to \mathbb{R}$ are all nondegenerate. Lemma 2.4 implies that critical points of energy greater 4 have index at least $n$.

Straightforward adaptations[1] of [3, Lemma 4.11 (1)] or of [9, 2.5.16 Theorem] imply that $\bar{f}$ is homotopic to a map $f$ with image in $\Lambda^{4.1}(x,y)$.

The map $g = \tau f$ has the desired energy bound since, for each $\theta \in S^{n-1}$,

$$E(g(\theta)) = E(\tau * f(\theta)) = \frac{E(f(\theta))}{1-\epsilon} + \epsilon < 2E(f(\theta)) + \frac{1}{2} < 9.$$

□

## References


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[1] Succinctly, the finitely many critical points of $E$ with energy between 4 and the maximal energy of a curve in the image of $\bar{f}$ are the only possible obstructions to deforming $\bar{f}$ to the desired map $f$ via the flow of a gradient like vector field for $-E$; such critical points can be bypassed as a consequence of transversality and the Morse Index Lemma.