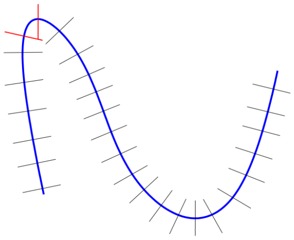

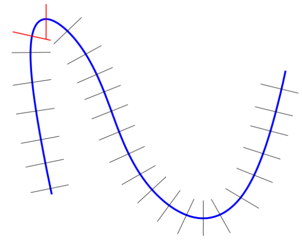

Mth829

Solutions to HW III

5) *Find the following limits*

a) $\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x}$.

Note that

$$f(x) = \begin{cases} \frac{1}{x}\log(1+x) & x > -1,\ x \neq 0 \\ 1 & x = 0 \end{cases}$$

is continuous (since $\ln(1) = 0$ and $d\log(1+x)/dx = 1/1+x = 1$ at $x = 0$). Furthermore, from the power series for log,

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n-1}x^n, \quad |x| < 1,$$

we see that f has a convergent power series,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(-x)^{n-1},$$

for $|x| < 1$. In particular, f is differentiable. However the limit to be computed is just

$$-\left.\frac{d}{dx}e^{f(x)}\right|_{x=0} = -e^{f(0)}f'(0) = \frac{e}{2}.$$

b) $\lim_{n\to\infty} \frac{n}{\log n}[n^{1/n} - 1]$

Since $n^{1/n} = e^{(\log n)/n}$, and $(\log n)/n \to 0$ as $n \to \infty$, what we are looking at is just

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \left.\frac{d}{dx}e^x\right|_{x=0} = 1.$$

c) $\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}$

Let's use the power series for $\sin x$ and $\cos x$. First write

$$\frac{\tan x - x}{x(1-\cos x)} = \frac{\frac{\sin x}{x} - \cos x}{\cos x(1-\cos x)}.$$

Since $\cos x \to 1$ as $x \to 0$ we may equally well consider the limit

$$\lim_{x\to 0} \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x}.$$

Now

$$\sin x = x - \frac{1}{6}x^3 + \cdots = x\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n}$$

while

$$\cos x = 1 - \frac{1}{2}x^2 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n}.$$

Thus

$$\frac{\sin x}{x} - \cos x = \sum_{n=1}^{\infty}\left[(-1)^n\frac{1}{(2n+1)!} - (-1)^n\frac{1}{(2n)!}\right]x^{2n} = x^2\left[\frac{1}{3} + \sum_{n=2}^{\infty}(-1)^n\left[\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right]x^{2n-2}\right]$$

and

$$1 - \cos x = x^2\left[\frac{1}{2} - \sum_{n=2}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n-2}\right].$$

Thus

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\frac{\sin x}{x} - \cos x}{1-\cos x} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

d) $\lim_{x\to 0}\frac{x-\sin x}{\tan x - x}$ This is very similar to (c).

6) *Suppose $f(x)f(y) = f(x+y)$ for all real x and y.* **a)** *Assuming f is differentiable and not zero, prove that*

$$f(x) = e^{cx}$$

where c is constant.

Note that $f(0) = 1$, since $f(x) = f(x)f(0)$ and $f(x) \neq 0$ for some x. Since f is differentiable,

$$c = f'(0)$$

exists. For $x \neq 0$ we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} f(x)\frac{f(h)-1}{h} = f(x)f'(0) = cf(x).$$

Thus $f'(x) = cf(x)$. It follows that $\frac{d}{dx}e^{-cx}f(x) = 0$ so

$$f(x) = f(0)e^{cx} = e^{cx}.$$

b) *Prove the same thing, assuming only that f is continuous.*

This is trickier. Note that we still have $f(0) = 1$. Also $f(x) > 0$ for all x since $f(x) = f(x/2)^2$. Let

$$c = \log f(1).$$

It follows that

$$f(n) = e^{nc},$$

for all $n \in \mathbb{Z}$. Also for $n, m \in \mathbb{Z}$, $m \neq 0$, we have

$$f(n/m)^m = f(n) = e^{nc},$$

so

$$f(n/m) = e^{nc/m}$$

since $f(n/m) \geq 0$. Thus

$$f(x) = e^{cx} \quad x \in \mathbb{Q}.$$

By continuity, we see that $f(x) = e^{cx}$ for all $x \in \mathbb{R}$.

9) a) *Put* $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$. *Prove that*

$$\lim_{N\to\infty} (s_N - \log N)$$

exists.

Write

$$\log N = \sum_{k=1}^{N-1} \log k + 1 - \log k = \sum_{k=1}^{N-1} \log\left(1 + \frac{1}{k}\right).$$

For each $0 < x \leq 1$, the series

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$$

converges. Thus for all k,

$$\frac{1}{k} - \log\left(1 + \frac{1}{k}\right) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n} \frac{1}{k^n}.$$

Since the series is alternating with terms of decreasing magnitude,

$$0 \leq \frac{1}{k} - \log\left(1 + \frac{1}{k}\right) \leq \frac{1}{2k^2}.$$

Thus

$$S_N - \log N = \frac{1}{N} + \sum_{k=1}^{N-1} \left[\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right],$$

where the sum is a sum of positive terms bounded in magnitude by $\sum_{k=1}^{N-1} 1/(2k^2)$ and thus converges.

b) *Roughly how large must* m *be so that* $N = 10^m$ *satisfies* $s_N > 100$*?*

To get $S_N > 100$ more or less we should have $\log N > 100$ so $m \log 10 > 100$. Since $\log 10 \approx 2.3$ (*log* denotes the natural log) we should have m roughly $10/2.3 \approx 4$ or larger. *We have to some between 10, 000 and 100,000 terms of the harmonic series to get over 100. The moral is,* $\sum \frac{1}{n}$ *diverges but very very slowly.*

12) *Suppose* $0 < \delta < \pi$, $f(x) = 1$ *if* $|x| \leq \delta$, $f(x) = 0$ *if* $\delta < |x| \leq 2\pi$, *and* $f(x+2\pi) = f(x)$ *for all* x.

a) *Compute the Fourier coefficients of f.*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx = \begin{cases} \frac{\delta}{\pi} & \text{if } n = 0, \\ \frac{\sin(n\delta)}{n\pi} & \text{if } n \neq 0. \end{cases}$$

b) *Conclude that* $\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2} \quad (0 < \delta < \pi)$.
Since $f(x)$ is Lipschitz continuous at 0, Thm. 8.14 applies and we see that

$$1 = \sum_{n=-\infty}^{\infty} c_n = \frac{\delta}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin(n\delta) - \sin(-n\delta)\right] = \frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n}.$$

The claimed identity follows.

c) *Deduce from Parseval's theorem that*

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}.$$

By Parseval,

$$\frac{\delta}{\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{n=-\infty}^{\infty} c_n^2 = \frac{\delta^2}{\pi^2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2}.$$

Thus

$$\pi - \delta = 2\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta}.$$

d) *Let $\delta \to 0$ and prove that*

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

The sum in part (c) can be understood as a Riemann sum for the integral indicated, with the function evaluated at the right endpoints of the intervals $[(n-1)\delta, n\delta]$,

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta^2}[n\delta - (n-1)\delta].$$

The result follows.

e) *Put $\delta = \pi/2$ in (c). What do you get?*
You get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

since $\sin^2(n\pi/2) = 1$ if n is odd and $= 0$ if n is even.

13) *Put $f(x) = x$ is $0 \le x < 2\pi$ and apply Parseval's theorem to conclude that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Compute the Fourier coefficients,

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx}.$$

We get

$$c_0 = \pi$$

, and for $n \neq 0$ we can integrate by parts

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x \left(-\frac{1}{in} \frac{d}{dx} e^{-inx} \right) dx = -\frac{1}{in} e^{-2\pi i n} + \frac{1}{2\pi i n} \int_0^{2\pi} e^{-inx} dx = \frac{i}{n}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{|c_0|^2}{2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} |c_n|^2 = -\frac{\pi^2}{2} + \frac{1}{4\pi} \int_0^{2\pi} x^2 dx = -\frac{\pi^2}{2} + \frac{2\pi^2}{3} = \frac{\pi^2}{6}.$$

14) *If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that*

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

First suppose the indicated series for f converges at all x. Then plugging in $x = 0$ gives

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2},$$

so

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Furthermore, computing the integral of $f(x)^2$ we see that

$$\frac{\pi^4}{5} = \frac{1}{\pi} \int_0^{\pi} (\pi - x)^4 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 dx = \frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \frac{1}{2},$$

since $\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(nx) dx = \frac{1}{2}$ and $\int_{-\pi}^{\pi} \cos(nx)\cos(mx) dx = 0$ if $n \neq m$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8} \left[\frac{1}{5} - \frac{1}{9} \right] \pi^4 = \frac{\pi^4}{90}.$$

So it remains to show that the series holds for f. Since f is Lipschitz continuous everywhere, this amounts to computing the Fourier coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \cos(nx).$$

Thus

$$c_0 = \frac{\pi^2}{3},$$

and, by integration by parts,

$$c_n = \frac{2}{\pi n} \int_0^{\pi} (\pi - x) \sin(nx) dx, \quad n \neq 0.$$

By a further, integration by parts,

$$c_n = \frac{2}{n^2} - \frac{2}{\pi n^2} \int_0^{\pi} \cos(nx) dx = \frac{2}{n^2}, \quad n \neq 0.$$

Thus

$$(\pi - |x|)^2 = f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (e^{inx} + e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx).$$