Mth829

Comments on HW I

Problems 1, 2, and 3 didn't cause many difficulties.

4) Most of you got the main idea that the absolute convergence of the series can be obtained by comparing to $\sum_n \frac{1}{n^2}$. A number of you forgot to observe that the series does not converge for $x = -1/n^2$ simply because one of the terms is ill defined there! As for uniform convergence, the most succinct answer would have been: "The series converges uniformly for x restricted to any compact interval in $\mathbb{R} \setminus (\{0\} \cup \{-1/n^2 \ : \ n = 1, 2, 3, \ldots\})$" together with a proof of that statement. Most of you stated some part of this, but the interval $(-1, 0)$ seemed to cause some problems. Go back and look at the problem and try to show that on any interval $[a, b]$ with $-1/m^2 < a < b < -1/(m+1)^2$ you have uniform convergence. It follows from the uniform convergence that you have continuity. The function is not bounded, since it diverges as you approach 0 or $-1/m^2$ for some m.

6) Most everyone got the point that the series is not absolutely convergent by comparing the absolute values to $1/n$. To show the uniform convergence on compact intervals, some of you tried splitting the sum

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

The problem with this is that is in general *not true* that

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \qquad (\star)$$

if the series converge conditionally. What *is* true, and what would make this work is the observation that *if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent and* $\sum_{n=1}^{\infty} b_n$ *is convergent then* $\sum_{n=1}^{\infty} (a_n + b_n)$ *is convergent and* $(\star)$ *holds.* (Try to prove this.) With this observation the splitting above is justified and since the first series converges uniformly on a compact interval (by the M test, say) and the second series is independent of x but conditionally convergent we get the result.

Another approach is to consider the series $\sum_n (-1)^n (x^2 + n)/n^2$ itself as an alternating series, noting that

$$\frac{x^2 + n}{n^2} > \frac{x^2 + (n+1)}{(n+1)^2}.$$

This shows conditional convergence and that the partial sums satisfy

$$\left| \sum_{k=1}^{m} (-1)^k \frac{x^2 + k}{k^2} - \sum_{k=1}^{n} (-1)^k \frac{x^2 + k}{k^2} \right| \leq \left| \sum_{k=m+1}^{n} (-1)^k \frac{x^2 + k}{k^2} \right| \leq \frac{x^2 + m + 1}{(m+1)^2},$$

if $m < n$, say. Since the right hand side converges to 0 uniformly for x in compact sets —

$$\sup_{|x| \leq R} \frac{x^2 + m + 1}{(m+1)^2} = \frac{R^2}{(m+1)^2} + \frac{1}{m+1}$$

— uniform convergence on compact sets results.

8) This problem caused some difficulties. Most of you saw that the series converged uniformly by the M test. However, proving continuity for $x \neq x_n$ proved to be a problem. Some of you observed, correctly, that the functions $I[x - x_n]$ are continuous on $A = (a,b) \setminus \{x_n : n = 1, 2, \ldots\}$ and that it follows that the series, which is a uniform limit of a some of these, is continuous on A. The difficulty is that you now have to show that the function $f : (a,b) \to \mathbb{R}$ which is the sum of the series is still continuous at the points of A. We know that

$$\lim_{\substack{y \to x \\ y \in A}} f(y) = f(x)$$

if $x \in A$, but we need to verify that

$$\lim_{y \to x} f(y) = f(x)$$

if $y \to x$ through points in (a,b). The difficulty is that $\{x_n\}$ might be, for instance, *dense in* (a,b) — think of the rational numbers in (a,b). So how do we do this? Here is an argument. Fix $x \in A$ and $\epsilon > 0$. By convergence of $\sum_n |c_n|$ there is an n such that

$$\sum_{k=n}^{\infty} |c_k| < \epsilon.$$

Now let $\delta = \min\{|x - x_k| \ : \ k = 1, \ldots, n-1\} > 0$. (The minimum is finite since the set is finite and $x \neq x_k$ for any k.) It follows for y with $|y - x| < \delta$ that

$$I(y - x_k) = I(x - x_k) \quad k = 1, \ldots, n-1.$$

Thus, for $|y - x| < \delta$

$$|f(y) - f(x)| \leq \sum_k |c_k||I(y - x_k) - I(x - x_k)| = \sum_{k=n}^{\infty} |c_k||I(y - x_k) - I(x - x_k)| \leq \sum_{k=n}^{\infty} |c_k| < \epsilon.$$

Notice that this argument works even if there is a subsequence $x_{n_k} \to x$, as would happen if $\{x_n\}$ were dense in (a,b). By the way, the function f is also discontinuous at each point x_k, with a jump of size c_k.

Problems 9 and 11 did not cause any serious difficulties, although I would have liked to see a more thorough explanation of the "summation by parts" argument in problem 11, namely why is

$$\sum_{n=1}^{N} f_n g_n = \sum_{n=1}^{N} F_n (g_n - g_{n+1}) + F_N g_{N+1},$$

with $F_n = \sum_{k=1}^{n} f_k$?

Problem 13 we discussed fairly thoroughly in class last friday.