The protean chromatic polynomial

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Universidad de Sevilla
October 2021

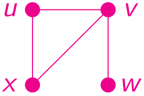
"Protean" means changeable. From Proteus, a Greek god of the sea and water.

"Chromatic" means having to do with color.

Proteus

A colorful annulus

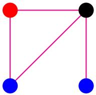
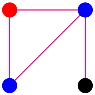
Let G be a finite graph with vertices V and edges E.

Ex. $G =$ 

$V = \{u, v, w, x\}$
$E = \{uv, ux, vw, vx\}$

A *coloring* of G is a function $c : V \to \{c_1, \ldots, c_t\}$. The coloring is *proper* if

$$uv \in E \implies c(u) \neq c(v).$$

Ex.  proper,  not proper, $\chi(G) = 3$.

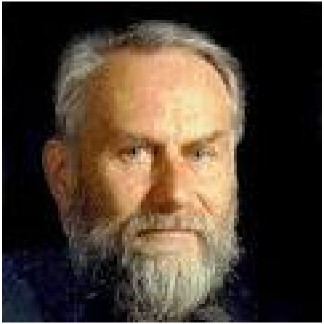
The *chromatic number* of G is

$\chi(G) =$ smallest t such that there is a proper $c : V \to \{c_1, \ldots, c_t\}$.

Theorem (Four Color Theorem, Appel-Haken, 1976)

If G is planar (can be drawn in the plane without edge crossings) then $\chi(G) \leq 4$. $\square$

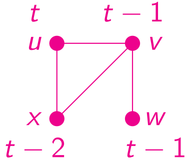

Kenneth Appel


Wolfgang Haken

For a positive integer t, the *chromatic polynomial* of G is

$$P(G) = P(G,t) = \# \text{ of proper colorings } c : V \to \{c_1, \ldots, c_t\}.$$

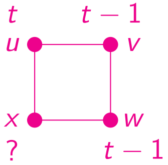
Ex. Coloring vertices in the order u, v, w, x gives choices



$$\begin{aligned} P(G,t) &= t(t-1)(t-1)(t-2) \\ &= t^4 - 4t^3 + 5t^2 - 2t \end{aligned}$$

Note 1. This is a polynomial in t.
2. $\chi(G)$ is the smallest positive integer with $P(G, \chi(G)) > 0$.
3. $P(G,t)$ need not be a product of factors $t-k$ for integers k.

Ex. Coloring vertices in the order u, v, w, x gives choices



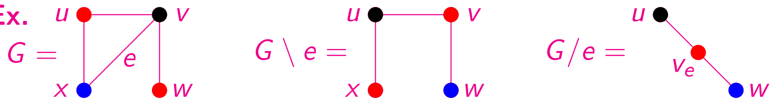
If $G = (V, E)$ is a graph and $e \in E$ then let

$$G \setminus e = G \text{ with } e \text{ deleted.}$$

$$G/e = G \text{ with } e \text{ contracted to a vertex } v_e.$$

Any multiple edge in G/e is replaced by a single edge.

Ex.



Lemma (Deletion-Contraction, DC)

If $G = (V, E)$ is any graph and $e \in E$ then

$$P(G, t) = P(G \setminus e; t) - P(G/e; t).$$

Proof.

Let $e = vx$. It suffices to show $P(G \setminus e) = P(G) + P(G/e)$.

$$\begin{aligned} P(G \setminus e) &= (\# \text{ of proper } c : G \setminus e \to \{c_1, \dots, c_t\} \text{ with } c(v) \neq c(x)) \\ &\quad + (\# \text{ of proper } c : G \setminus e \to \{c_1, \dots, c_t\} \text{ with } c(v) = c(x)) \\ &= P(G) + P(G/e) \end{aligned}$$

as desired. $\square$

$$P(G,t) = P(G \setminus e; t) - P(G/e; t).$$

Theorem (Birkhoff, 1912)

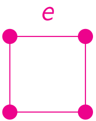
For any graph $G = (V, E)$, $P(G, t)$ is a polynomial in t.

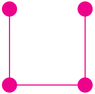
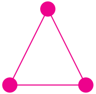
Proof.

Let $|V| = n, |E| = m$. Induct on m. If $m = 0$ then $P(G) = t^n$.
If $m > 0$, then pick $e \in E$. Both $G \setminus e$ and G/e have fewer edges than G. So by DC and induction

$$P(G) = P(G \setminus e) - P(G/e) = \text{polynomial} - \text{polynomial} = \text{polynomial}$$

as desired. $\square$

Ex. $P\Bigg($ 

$\Bigg) = P\Bigg($  $\Bigg) - P\Bigg($  $\Bigg)$

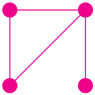
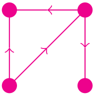
$$= t(t-1)^3 - t(t-1)(t-2)$$

$$= t(t-1)(t^2 - 3t + 3).$$


George David Birkhoff

An *orientation* of graph $G = (V, E)$ is a directed graph O obtained by replacing each $uv \in E$ by one of the arcs $\vec{uv}$ or $\vec{vu}$. So the number of orientations of G is $2^{|E|}$. A *directed cycle* of O is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ with $\vec{v_i v_{i+1}}$ an arc for all i modulo k. Orientation O is *acyclic* if it has no directed cycles.

Ex.

$G =$  has orientation $O =$  which is acyclic.

$\#$ of acyclic orientations of G

$= (\#$ for the triangle$)(\#$ for the remaining edge$)$

$= (2^3 - 2)(2) = 12.$

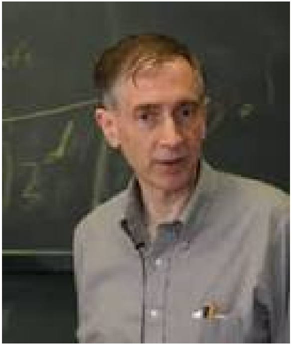
$$P(G, -1) = (-1)^4 - 4(-1)^3 + 5(-1)^2 - 2(-1) = 12.$$

Theorem (Stanley, 1973)

For any graph G with $|V| = n$,

$$P(G, -1) = (-1)^n (\#\text{ of acyclic orientations of } G). \qquad \square$$

Note: Blass and S (1986) gave a bijective proof of this theorem.

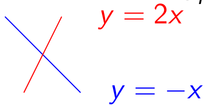


Richard P. Stanley

Andreas Blass

A *hyperplane* in $\mathbb{R}^n$ is a subspace H with $\dim H = n-1$. A *hyperplane arrangement* is a set of hyperplanes $\mathcal{A} = \{H_1, \dots, H_k\}$. The *regions* of $\mathcal{A}$ are the connected components of $\mathbb{R}^n - \cup_i H_i$.

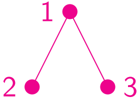
Ex. $\mathcal{A} = \{y = 2x,\ y = -x\} \subset \mathbb{R}^2$.
$\#$ of regions of $\mathcal{A} = 4$.



Let $[n] = \{1, 2, \dots, n\}$. Graph G with $V = [n]$ has arrangement

$$\mathcal{A}(G) = \{x_i = x_j \ : \ ij \in E\}.$$

Ex. $G =$



$\mathcal{A}(G) = \{x_1 = x_2,\ x_1 = x_3\} \subset \mathbb{R}^3$.
$\#$ of regions of $\mathcal{A}(G) = 4$.

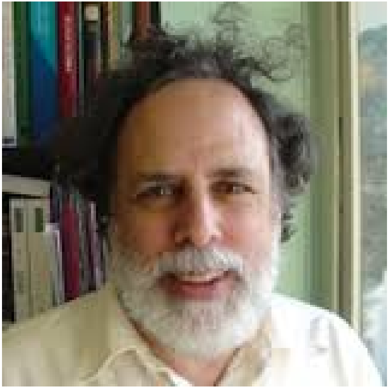
$P(G, t) = t(t-1)^2 \implies P(G, -1) = -(-2)^2 = -4$.

Theorem (Zaslavsky, 1975)

For any graph G with $V = [n]$,

$$P(G, -1) = (-1)^n (\#\text{ of regions of } \mathcal{A}(G)). \qquad \square$$

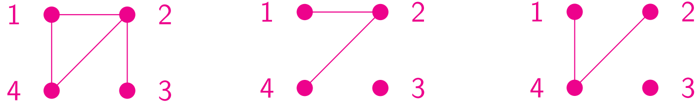
There is a bijection: acyclic orientations of $G \leftrightarrow$ regions of $\mathcal{A}(G)$.



Thomas Zaslavsky

Let G be a graph with $V = [n]$ and F be a spanning forest (an acyclic subgraph using all the vertices of G). Then F is *increasing* if the vertices on any path of F starting at the minimum vertex of its connected component form an increasing sequence.

Ex.



G $\qquad$ F increasing $\qquad$ F not increasing

Define

$$\mathrm{isf}_m(G) = \#\text{ of increasing spanning forests of } G \text{ with } m \text{ edges.}$$

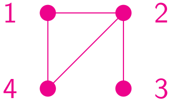
and

$$\mathrm{ISF}(G, t) = \sum_{m=0}^{n-1} (-1)^m \, \mathrm{isf}_m(G) t^{n-m}.$$

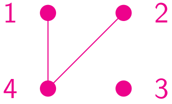
$\mathrm{isf}_m(G) = \#$ of increasing spanning forests of G with m edges.

$$\mathrm{ISF}(G) = \mathrm{ISF}(G, t) = \sum_{m \geq 0} (-1)^m \,\mathrm{isf}_m(G) t^{n-m}.$$

Ex. $G =$



not increasing:

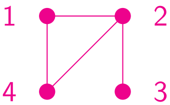


$$\begin{aligned}
\mathrm{isf}_0(G) &= 1\\
\mathrm{isf}_1(G) &= |E| = 4\\
\mathrm{isf}_2(G) &= \binom{4}{2} - 1 = 5\\
\mathrm{isf}_3(G) &= \binom{4}{3} - 2 = 2\\
\mathrm{isf}_4(G) &= 0\\
\mathrm{ISF}(G) &= t^4 - 4t^3 + 5t^2 - 2t = t(t-1)^2(t-2).
\end{aligned}$$

Let G be a graph with vertex set $V = [n]$. For $j \in [n]$ define

$$V_j = \{i \in V \mid i < j \text{ and } ij \in E\}.$$

Ex.

$G =$ 

$V_1 = \emptyset$,
$V_2 = \{1\}$ because $12 \in E$,
$V_3 = \{2\}$ because $23 \in E$,
$V_4 = \{1, 2\}$ because $14, 24 \in E$
and

$$(t - |V_1|)(t - |V_2|)(t - |V_3|)(t - |V_4|) = t(t-1)^2(t-2) = \mathrm{ISF}(G).$$

Theorem (Hallam-S, 2014)

Let G have $V = [n]$ and V_j as defined above. Then

$$\mathrm{ISF}(G, t) = \prod_{j=1}^{n} (t - |V_j|).$$


Joshua Hallam

$$V_j = \{i \in V \mid i < j \text{ and } ij \in E\}.$$

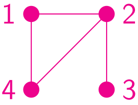
When is $\mathrm{ISF}(G,t) = P(G,t)$?

Theorem (Hallam-S, 2014)

Let G be a graph with $V = [n]$. Then $P(G,t) = \mathrm{ISF}(G,t)$ if and only if the induced subgraphs $G[V_j]$ are cliques for $1 \le j \le n$. $\square$

Ex.

$G =$



$G[V_1] = G[\emptyset]$ $G[V_2] = G[1]$ $G[V_3] = G[2]$ $G[V_4] = G[1,2]$

This clique condition is called a *perfect elimination order*.

1. $P(G,t)$ **for negative** t**.** Let G be a graph with acyclic orientation O. Let t be a positive integer and $c : V \to \{1,2,\ldots,t\}$ be a coloring. Call (O,c) *compatible* if

$$\vec{uv} \in O \implies c(u) \le c(v).$$

Theorem (Stanley, 1973)

Let G have $|V| = n$ and let t be a positive integer. Then

$$P(G,-t) = (-1)^n(\#\text{ of compatible pairs }(O,c)). \qquad \square$$

2. Arbitrary hyperplanes. Any hyperplane arrangement $\mathcal{A}$ has an associated characteristic polynomial $\mathrm{ch}(\mathcal{A},t)$. If $\mathcal{A} = \mathcal{A}(G)$ then $\mathrm{ch}(\mathcal{A},t) = P(G,t)$.

Theorem (Zaslavsky, 1975)

For any hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$

$$\mathrm{ch}(\mathcal{A},-1) = (-1)^n(\#\text{ of regions of }\mathcal{A}). \qquad \square$$

3. Symmetric functions. Consider variables $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$. A power series in $\mathbf{x}$ is *symmetric* if it is invariant all under permutations of variables. The *chromatic symmetric function* of G with $V = \{v_1, \dots, v_n\}$ is

$$X(G) = X(G, \mathbf{x}) = \sum_{c} x_{c(v_1)} \cdots x_{c(v_n)}$$

where the sum is over all proper colorings $c : V \to \mathbb{Z}^+$. This is a symmetric function in the variables $\mathbf{x}$. Setting $x_1 = \cdots = x_t = 1$ and $x_i = 0$ for $i > t$ gives $X(G; \mathbf{x}) = P(G; t)$.

Theorem (Stanley, 1995)

Let e_λ be the elementary symmetric function corresponding to λ. If $X(G) = \sum_\lambda c_\lambda e_\lambda$, then

$$\#\text{ of acyclic orientations of } G \text{ with } k \text{ sinks} = \sum_{\ell(\lambda)=k} c_\lambda.$$

where ℓ is the length function. □

Unfortunately, $X(G)$ does not satisfy a DC Lemma. Using symmetric functions in noncommuting variables (Rosas-S) one can derive such a lemma (Gebhard-S).

Mercedes Rosas

David Gebhard

Jeremy Martin

4. More on increasing forests. Hallam, Martin, and S (2018) have extended these results to simplicial complexes of arbitrary dimension d. When $d = 1$ one recovers the theorems for graphs.

5. Log concavity. A polyomial $P(t) = a_0 + a_1 t + \cdots + a_n t^n$ is *log concave* if

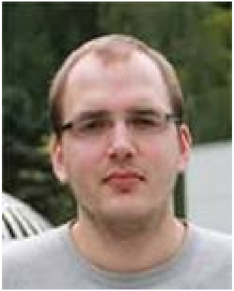
$$a_k^2 \geq a_{k-1} a_{k+1}$$

for all $0 < k < n$. Using deep methods from algebraic geometry (Chern classes, etc.), Huh has proven the following.

Theorem (Huh, 2013)

For any graph G, the polynomial $P(G, t)$ is log concave. □

Adiprasito, Huh, and Katz gave a combinatorial Hodge Theory proof of the Heron-Rota-Welsh conjecture which generalizes Huh's result to any matroid.



Karim Adiprasito

June Huh

Eric Katz

¡GRACIAS POR ESCUCHAR!