Distance Regular Graphs and Unimodality

John S. Caughman, IV
and
Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027

1. Distance regular graphs

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Distance Regular Graphs

Let $\Gamma = (X, R)$ be a graph with vertices X and edges R (relations). Let ∂ be Γ's distance function and define the *sphere of radius i about $x \in X$* to be

$$\Gamma_i(x) = \{y \in X \ : \ \partial(x, y) = i\}.$$

Γ is a *distance regular graph* or *drg* if, given x, y with $\partial(x, y) = h$, the cardinality

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

depends only on i, j and h and not on the particular x, y chosen. Thus the *valencey*

$$k_i := |\Gamma_i(x)| = p^0_{ii}$$

does not depend on x. (So Γ is regular of degree $k := k_1$.)

A sequence of real numbers is *unimodal* if for some index m

$$a_0 \le a_1 \le \ldots \le a_m \ge a_{m+1} \ge \cdots$$

Theorem 1 *Let Γ be a drg of diameter D. Then the sequence*

$$k_0, k_1, \ldots, k_D$$

is unimodal. ■

For example, the D-cube is a drg with $k_i = \binom{D}{i}$ and

$$\binom{D}{0} \leq \binom{D}{1} \leq \ldots \leq \binom{D}{\lfloor D/2 \rfloor} \geq \ldots \geq \binom{D}{D}.$$

The *ith distance matrix*, A_i, has rows and columns indexed by X and

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{else.} \end{cases}$$

So $A := A_1$ is the usual adjacency matrix.

Let J be the $|X| \times |X|$ matrix of all 1's. Then the matrices $A_0, \ldots, A_D$ are non-zero 0,1-matrices satisfying

1. $A_0 = I$,

2. $\sum_i A_i = J$,

3. $A_i^t = A_i$ for $0 \leq i \leq D$,

4. $A_i A_j = \sum_h p_{ij}^h A_h$ for $0 \leq i, j \leq D$,

5. A_i is a polynomial of degree i in A for $0 \leq i \leq D$.

Association Schemes

An *association scheme* is $Y = (X, \{A_0, \ldots, A_D\})$ such that the A_i are nonzero 0,1-matrices whose rows and columns are indexed by X such that

(1) $A_0 = I$,

(2) $\sum_i A_i = J$,

(3) $A_i^t = A_i$ for all i,

(4) $A_i A_j = \sum_h p_{ij}^h A_h$ for scalars p_{ij}^h.

Association scheme Y is *P-polynomial* if there is an indexing of the A_i such that

A_i is a polynomial of degree i in A for $0 \leq i \leq D$.

Theorem 2 *Y is P-polynomial if and only if it arises from a drg.* ■

The *Bose-Mesner algebra of Y* is the vector space

$$M = \text{span}\{A_0, \ldots, A_D\}.$$

Note that M is an algebra by (4) and that the set $\{A_0, \ldots, A_D\}$ is independent and so a basis by (2).

(3) and (4) show that the A_i are symmetric and commute and so simultaneously diagonalisable. Let $E_1, \ldots, E_D$ be the orthogonal projections onto the common eigenspaces. They're a basis for M and

(1') $E_0 = \frac{1}{|X|} J$,

(2') $\sum_i E_i = I$,

(3') $E_i^t = E_i$ for all i,

(4') $E_i \circ E_j = \sum_h q_{ij}^h E_h$ for scalars q_{ij}^h where $\circ$ is *Schur product* (entry-wise).

Note that $A_i \circ A_j = \delta_{ij} A_i$ and $E_i E_j = \delta_{ij} E_i$.

Y is *Q-polynomial* if for some indexing of the E_i we have E_i is a Schur polynomial of degree i in E_1, $\forall i$. The *multiplicities* of Y are $m_i := \operatorname{rk} E_i$.

Conjecture 3 (Bannai and Ito) *If Y is a Q-polynomial scheme then the sequence*

$$m_0, m_1, \ldots, m_D$$

is unimodal.

Theorem 4 (C & J) *If Y is a Q-polynomial scheme which is also dual-thin then*

$$m_i \leq m_{i+1} \quad \textit{and} \quad m_i \leq m_{D-i} \quad \textit{for} \quad i < D/2. \quad \blacksquare$$

Terwilliger Algebras

Element $x \in X$ has *ith dual idempotent* the diagonal matrix $E_i^* = E_i^*(x)$ such that

$$(E_i^*)_{yy} = (A_i)_{xy} \quad \text{for} \quad y \in X.$$

(1*) E_0^* has 1 in the xx position and 0 elsewhere,

(2*) $\sum_i E_i^* = I$,

(3*) $(E_i^*)^t = E_i^*$ for all i,

(4*) $E_i^* E_j^* = E_i^* \circ E_j^* = \delta_{ij} E_i^*$.

Note also that $\operatorname{rk} E_i^* = k_i$.

The *Terwilliger algebra*, $T(x)$ is the one generated by the E_i and E_j^*. It is a finite dimensional, semi-simple $\mathbb{C}$-algebra, not commutative in general. Let W be an irreducible $T(x)$-module in the decomposition of the standard module for $T(x)$. Then

$$W = \oplus_i E_i W.$$

Call W *dual thin* if for all i:

$$\dim(E_i W) \leq 1.$$

Finally Y is *dual thin* if every such $T(x)$-module is dual thin for all $x \in X$.