Möbius Functions of Posets III: Topology of Posets

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Outline

Shellability of Simplicial Complexes

The Euler Characteristic

The Order Complex

Lexicographic Shellability

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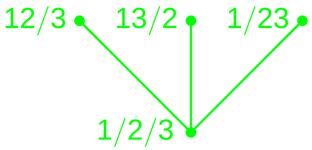

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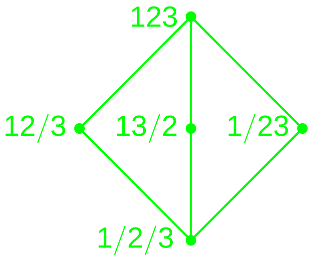

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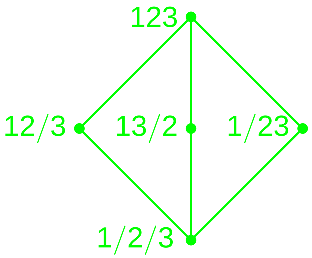

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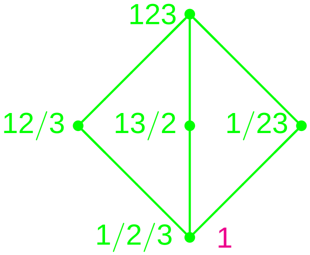

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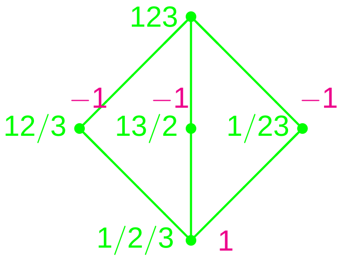

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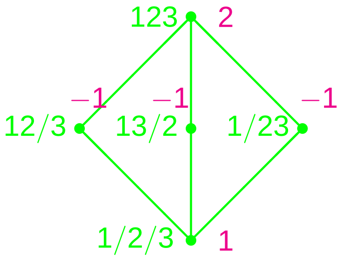

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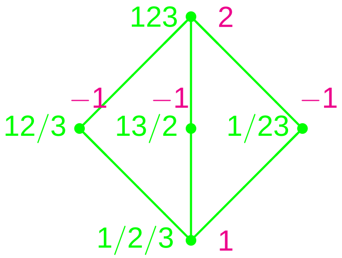

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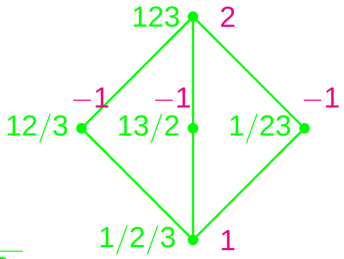

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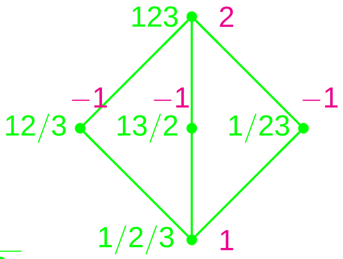

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Conjecture

We have: $\mu(\Pi_n) = (-1)^{n-1}(n-1)!.$

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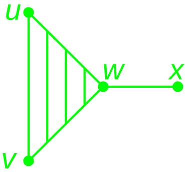

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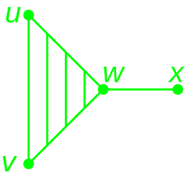

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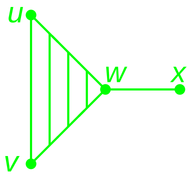

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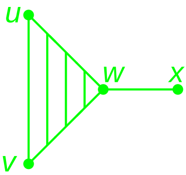

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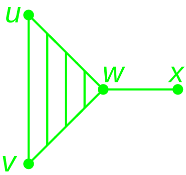

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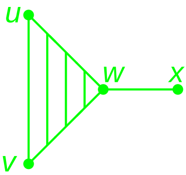

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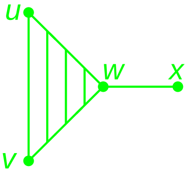

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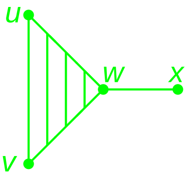

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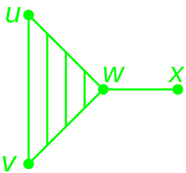

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Note. A simplicial complex pure of dimension 1 is just a graph.

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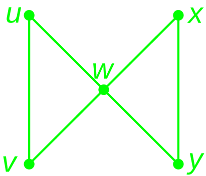

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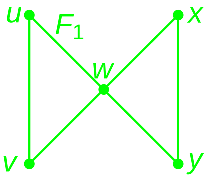

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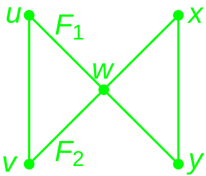

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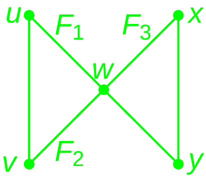

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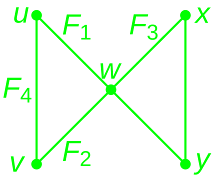

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Example. For the graph at right

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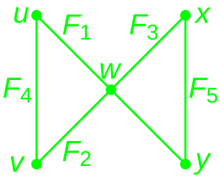

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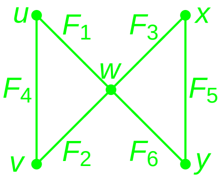

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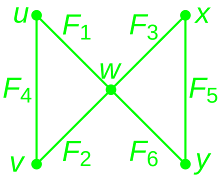

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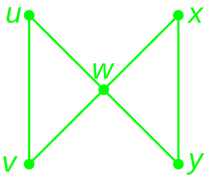

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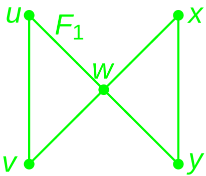

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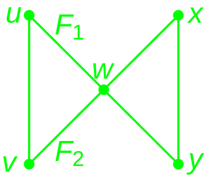

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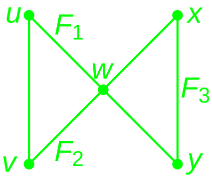

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Any sequence beginning $uw, \; vw, \; xy$
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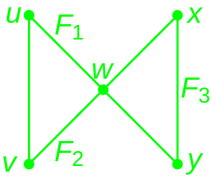

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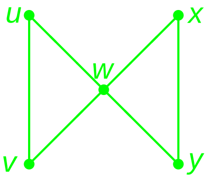

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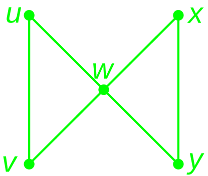

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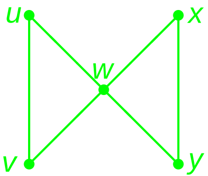

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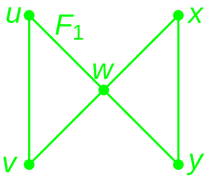

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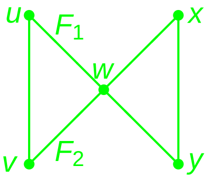

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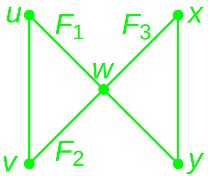

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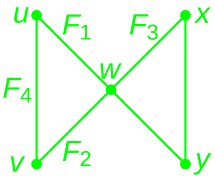

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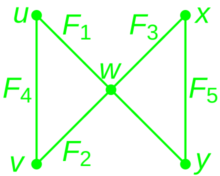

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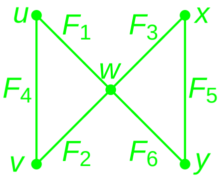

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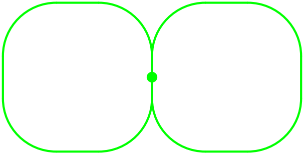
$\vee^2 S^1 =$ 

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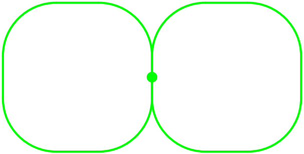
$\vee^2 S^1 =$ 

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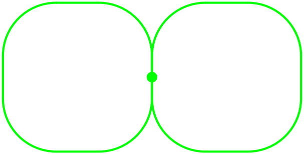
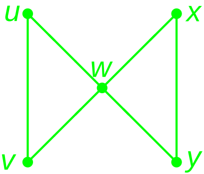
$\vee^2 S^1 =$  $\approx$ 

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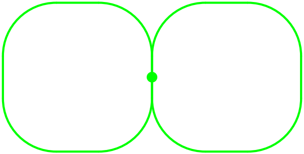
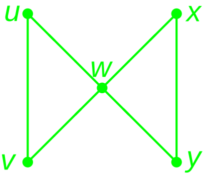
$\vee^2 S^1 =$  $\approx$ 

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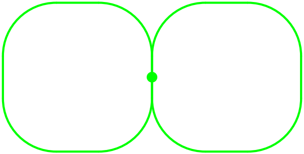
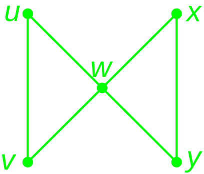
$\vee^2 S^1 =$  $\approx$ 

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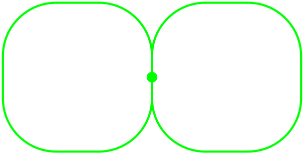
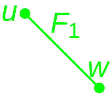
$\vee^2 S^1 =$  $\approx$ 

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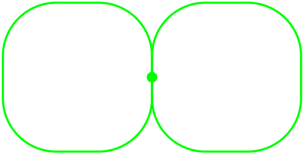

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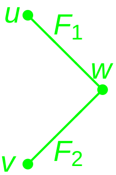

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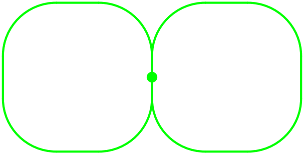
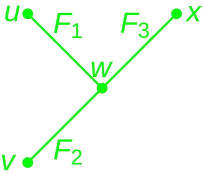
$\vee^2 S^1 =$  $\approx$ 

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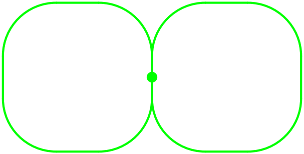

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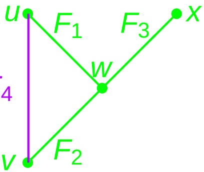

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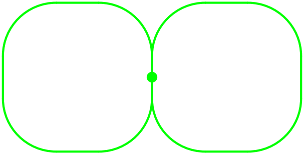
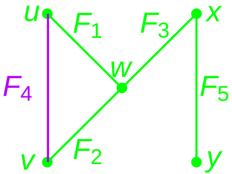
$\vee^2 S^1 =$  $\approx$ 

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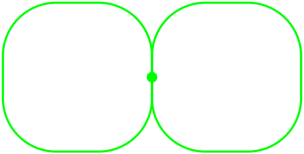
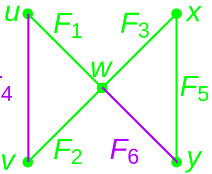
Let Δ be a shellable simplicial complex pure of dimension d. Then

$$\Delta \approx \vee^k S^d$$

where k is the number of facets satisfying $r(F) = F$ in a shelling of Δ.

Let S^d denote the d-sphere (sphere of dimension d). To form the *bouquet* or *wedge* of k spheres of dimension d, $\vee^k S^d$, take a point of each sphere and identify the points.

Example.

$\vee^2 S^1 =$  $\approx$ 

$r(uw) = \emptyset, \quad r(vw) = v, \quad r(wx) = x, \quad r(uv) = uv,$
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Outline

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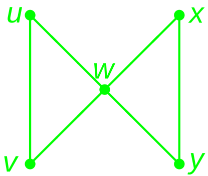

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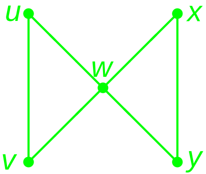

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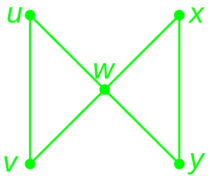

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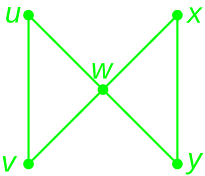

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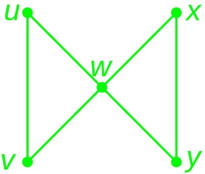

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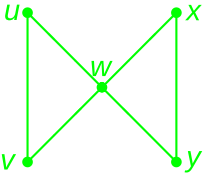

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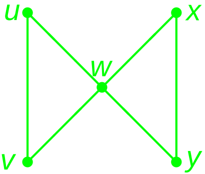

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Outline

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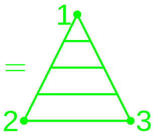
and $\Delta(C_4) =$ 

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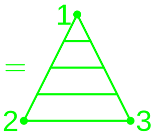
and $\Delta(C_4) =$ 

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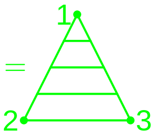

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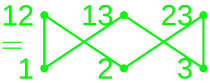

If $x, y \in P$ (poset) then an *x–y chain of length i in P* is a subposet $C : x = x_0 < x_1 < \ldots < x_i = y$. If P is bounded, let

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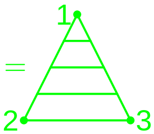

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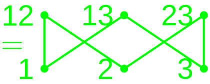

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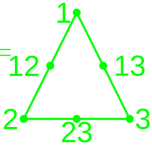

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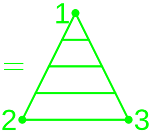

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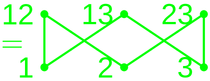

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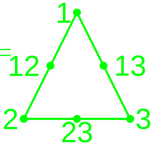

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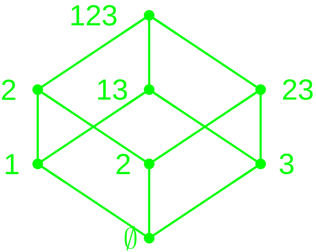
$B_3 =$ 

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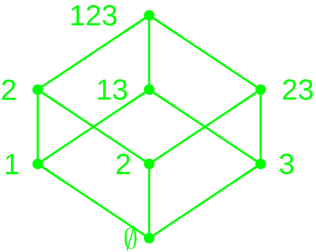

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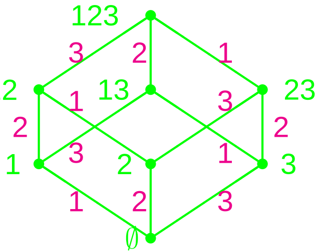

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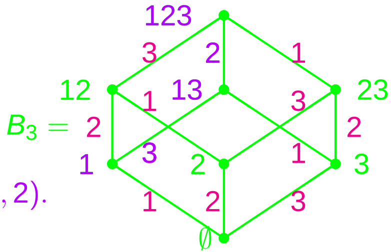
$$\ell(x_0 \lhd x_1 \lhd \ldots \lhd x_i) = (\ell(x_0 \lhd x_1), \ldots, \ell(x_{i-1} \lhd x_i)).$$

Example. For B_n

let $\ell(S \lhd T) = T - S$,
so $\ell(1 \lhd 13 \lhd 123) = (3, 2)$.



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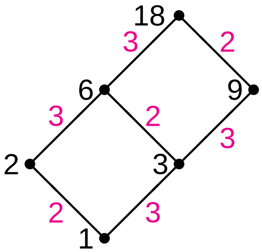

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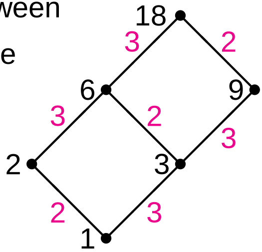

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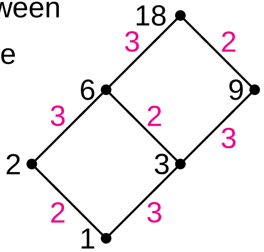

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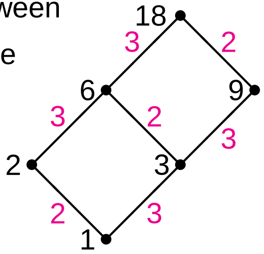

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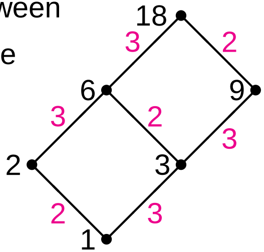

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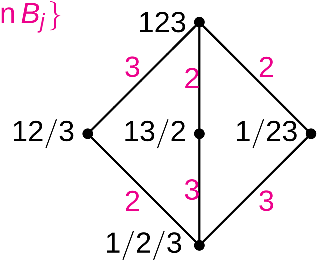

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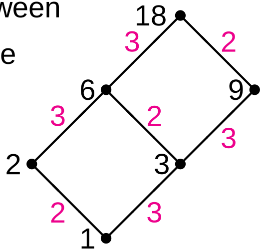

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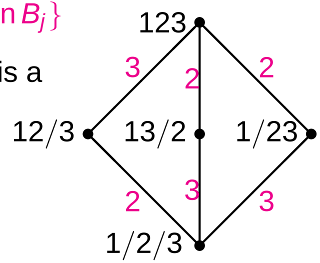

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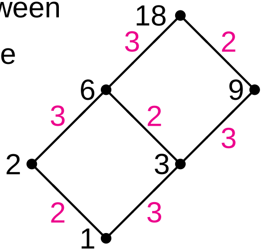

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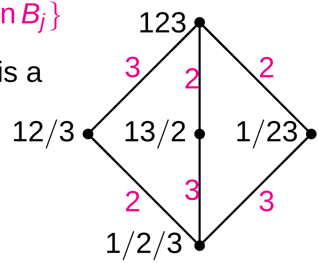

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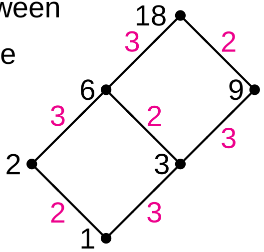

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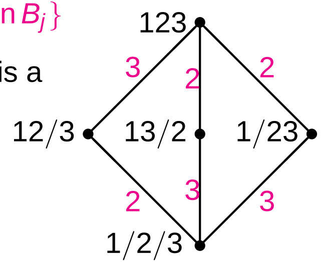

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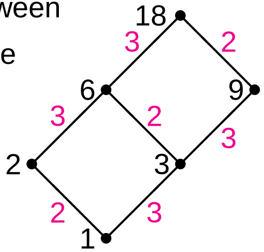

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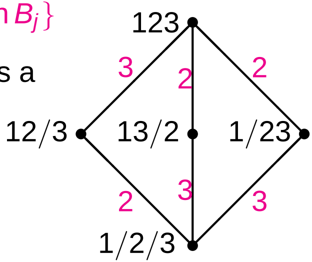

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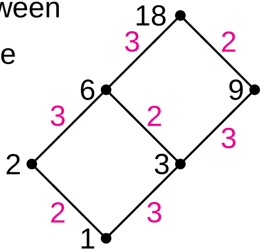

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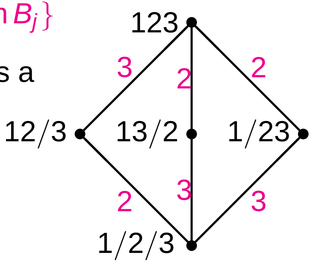

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Theorem (Björner, 1980)

Let P be a graded poset. If P has an EL-labelling then $\Delta(P)$ is shellable. In fact, if $F_1, \ldots, F_k$ is a list of the saturated $\hat{0} - \hat{1}$ chains in lexicographic order, then $\overline{F}_1, \ldots, \overline{F}_k$ is a shelling of $\Delta(P)$. Furthermore

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Corollary

(a) $\mu(C_n) = 0$ *if* $n \geq 2$.

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(c) $\mu(D_n) = \begin{cases} (-1)^k & \text{if } m_i = 1\ \forall i, \\ 0 & \text{if } \exists i \text{ with } m_i \geq 2, \end{cases}$ *where* $n = \prod_i p_i^{m_i}$.

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