Longest increasing subsequences and log concavity

Miklós Bóna
University of Florida

Marie-Louise Bruner
Technische Universität Wien

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

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The basic conjectures

Some results

Yet more conjectures

Let $\mathfrak{S}_n$ be the nth symmetric group and let $\pi = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$ be viewed as a sequence. Let

$$\ell(\pi) = \text{length of a longest increasing subsequence of } \pi.$$

Ex. If $\pi = 21435$ then a longest increasing subsequence is 245 so $\ell(\pi) = 3$.

Let

$$L_{n,k} = \{\pi \in \mathfrak{S}_n \mid \ell(\pi) = k\} \quad \text{and} \quad \ell_{n,k} = \#L_{n,k}.$$

Call a sequence of real numbers $c_1, c_2, \ldots, c_n$ *log concave* if

$$c_{k-1}c_{k+1} \le c_k^2 \text{ for all } 1 < k < n.$$

Conjecture (Chen, 2008)

For all $n \ge 1$, the following sequence is log concave:

$$\ell_{n,1}.\ell_{n,2}, \ldots, \ell_{n,n}.$$

Ex. If $n = 3$ then

$k=1$	$k=2$	$k=3$
321	132, 213, 231, 312	123
1	4	1

Let $\mathfrak{I}_n \subseteq \mathfrak{S}_n$ be the set of involutions in $\mathfrak{S}_n$. Also let

$$I_{n,k} = \{\pi \in \mathfrak{I}_n \mid \ell(\pi) = k\}$$

and

$$i_{n,k} = \# I_{n,k}.$$

Conjecture (BBS)

For all $n \geq 1$, the following sequence is log concave:

$$i_{n,1}.i_{n,2}, \ldots, i_{n,n}.$$

Ex. If $n = 3$ then

$k=1$	$k=2$	$k=3$
321	132, 213	123
1	2	1

Let RS denote the Robinson-Schensted map and $\operatorname{sh} P$ be the shape of a tableau P.

Theorem

If $RS(\pi) = (P, Q)$ then the following hold.

(1) $\operatorname{sh} P = \operatorname{sh} Q = (\lambda_1, \ldots, \lambda_t)$ with $\lambda_1 = \ell(\pi)$.

(2) We have $\pi \in \mathfrak{I}_n$ if and only if $P = Q$.

Because of (1), we can define the *shape of a permutation* to be

$$\operatorname{sh} \pi = \operatorname{sh} P = \operatorname{sh} Q$$

where $RS(\pi) = (P, Q)$. It will be convenient to define

$$\operatorname{sh}(\pi, \pi') = (\operatorname{sh} \pi, \operatorname{sh} \pi').$$

Because of (2), we can identify an involution with its tableau.

A map $f : I_{n,k-1} \times I_{n,k+1} \to I^2_{n,k}$ is called *shape preserving* (sp) if

$$\mathrm{sh}(\iota, \iota') = \mathrm{sh}(\kappa, \kappa') \quad \text{implies} \quad \mathrm{sh}\, f(\iota, \iota') = \mathrm{sh}\, f(\kappa, \kappa').$$

Theorem (BBS)

If there is an sp injection $f : I_{n,k-1} \times I_{n,k+1} \to I^2_{n,k}$ then there is an sp injection $F : L_{n,k-1} \times L_{n,k+1} \to L^2_{n,k}$.

Proof.

Define F as the composition of the maps

$$\begin{aligned}
(\pi, \pi') &\stackrel{RS^2}{\to} \big((P, Q),\ (P', Q')\big) \\
&\to \big((P, P'),\ (Q, Q')\big) \\
&\stackrel{f^2}{\to} \big((S, S'),\ (T, T')\big) \\
&\to \big((S, T),\ (S', T')\big) \\
&\stackrel{(RS^{-1})^2}{\to} \big(\sigma, \sigma'\big)
\end{aligned}$$

We used the fact that f is shape preserving in applying RS^{-1}. $\square$

Let

$$\ell_{n,k}^{\text{hook}} = \#\{\pi \in L_{n,k} \mid \operatorname{sh}\pi \text{ is a hook}\}$$

and

$$\ell_{n,k}^{\text{two-row}} = \#\{\pi \in L_{n,k} \mid \operatorname{sh}\pi \text{ has at most two rows}\}.$$

Similarly define $i_{n,k}^{\text{hook}}$ and $i_{n,k}^{\text{two-row}}$.

Proposition

The following sequences are all log concave for a given $n \geq 1$:

$$(\ell_{n,k}^{\text{hook}})_{1\leq k\leq n}, \quad (\ell_{n,k}^{\text{two-row}})_{1\leq k\leq n}, \quad (i_{n,k}^{\text{hook}})_{1\leq k\leq n}, \quad (i_{n,k}^{\text{two-row}})_{1\leq k\leq n}.$$

Proof.

The statements for involutions can be proved using the hook formula or combinatorially using Lindström-Gessel-Viennot. The statements for permutations now follow by applying arguments as in the previous theorem or using the fact that the entries in the permutation sequence are the squares of those in the corresponding involution sequence. □

(1) **Real roots.** Suppose the real sequence $c : c_0, c_1, \ldots, c_n$ has generating function (gf) $f(q) = c_0 + c_1 q + \ldots c_n q^n$. If the sequence is positive then $f(q)$ having only real roots implies the sequence is log concave. The gf's for the $\ell_{n,k}$ and $i_{n,k}$ are not real rooted, in general. Can anything nice be said about the roots?

(2) **Infinite log concavity.** The *L-operator* takes sequence $c : c_0, \ldots, c_n$ to sequence $L(c) : d_0, \ldots, d_n$ where

$$d_k = c_k^2 - c_{k-1} c_{k+1} \quad \text{with} \quad c_{-1} = c_{n+1} = 0.$$

Clearly c being log concave is equivalent to d being nonnegative. Call c *infinitely log concave* if $L^i(c)$ is nonnegative for all $i \geq 0$.

Conjecture (Chen)

For all $n \geq 1$, the following sequence is infinitely log concave:

$$\ell_{n,1}, \ell_{n,2}, \ldots, \ell_{n,n}.$$

Using a technique of McNamara and S we have been able to prove this for $n \leq 50$. It is *not* true that the involution sequence is infinitely log concave.

(3) q-**log convexity.** Define a partial order on polynomials with real coefficients by $f(q) \leq_q g(q)$ if $g(q) - f(q)$ has nonnegative coefficients. Call a sequence $f_1(q), f_2(q), \ldots$ *q-log convex* if

$$f_{n-1}(q) f_{n+1}(q) \geq_q f_n(q)^2 \text{ for all } n > 1.$$

Conjecture (Chen)

The sequence $\ell_1(q), \ell_2(q), \ldots$ is q-log convex where

$$\ell_n(q) = \ell_{n,1} q + \ell_{n,2} q^2 + \ldots \ell_{n,n} q^n.$$

This conjecture has been verified up through $n = 50$. The corresponding conjecture for involutions is false.

(4) **Perfect matchings.** A *perfect matching* is $\mu \in \mathfrak{I}_{2n}$ without fixed points. Chen has various conjectures for perfect matchings.

(5) **Limiting distribution.** As $n \to \infty$, the sequence $(\ell_{n,k})_{1 \leq k \leq n}$ approaches the Tracy-Widom distribution.

Theorem (Deift)

The Tracy-Widom distribution is log concave.

THANKS FOR LISTENING!