Distance-preserving graphs

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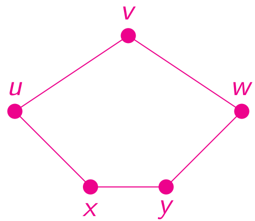
Let $G = (V, E)$ be a graph and let d or d_G denote its distance function. A subgraph $H \subseteq G$ is *isometric*, written $H \leq G$, if for every $u, v \in V(H)$ we have

$$d_H(u, v) = d_G(u, v)$$

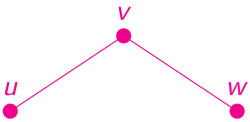
Ex. Consider

$G = C_5 =$

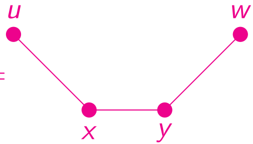


and

$H =$



$H' =$



Then $H \leq G$. But $H' \not\leq G$ since $d_{H'}(u, w) = 3$ and $d_G(u, w) = 2$.

Call a connected graph G *distance preserving* (dp) if it has an isometric subgraph with k vertices for all k with $1 \leq k \leq |V(G)|$.

Ex. From the previous example, C_5 is not dp since it has no isometric subgraph with 4 vertices. On the other hand, trees are dp: If T is a tree and v is a leaf then $T - v$ is an isometric subgraph of T. So by repeatedly removing leaves, one can find isometric subgraphs of T with any number of vertices.

Roughly, cycles cause obstructions to being dp.

Conjecture (Nussbaum-Esfahanian)

Almost all connected graphs are dp. That is, if d_n and c_n are the number of dp graphs and connected graphs on n vertices, respectively, then

$$\lim_{n\to\infty} \frac{d_n}{c_n} = 1.$$

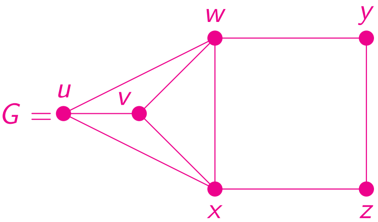
We will provide various techniques for constructing larger dp graphs from smaller ones.

The *neighborhood* of a vertex v of $G = (V, E)$ is

$$N(v) = \{w \mid vw \in E\}.$$

Call v *simplicial* if $N(v)$ is the vertex set of a clique (complete subgraph) of G.

Ex. Consider



Then u is simplicial since $N(u) = \{v, w, x\}$, the vertices of a triangle. But y is not simplicial since $N(y) = \{w, z\}$ and $wz \notin E$.

Theorem (Z)

Let v be simplicial in G. Then $G' = G - v$ is isometric in G.

Proof. Consider $x, y \in V(G')$. It suffices to show that no x–y geodesic (x–y path of minimum length) in G goes through v. Suppose, towards a contradiction, that there is such a geodesic

$$P : x = v_0, v_1, \ldots, v_s = v, \ldots, v_t = y.$$

Since v is simplicial $v_{s-1}v_{s+1} \in E(G)$. So $P - v$ is a shorter path from x to y, a contradiction. $\square$

A graph G is *chordal* if every cycle $C \subseteq G$ of length at least 4 has an edge of G joining two vertices not adjacent along C.

Corollary

Chordal graphs, and hence trees, are dp

Proof. If G is chordal, then it has a simplicial vertex v with $G - v$ chordal. The result now follows by induction. $\square$

Let G, H be graphs. Products of G and H have vertex set $V(G) \times V(H)$. Their *(Cartesian) product*, $G \square H$, has edge set

$$E(G \square H) = \{(a,x)(b,y) \mid x = y \ \& \ ab \in E(G), \text{ or } a = b \ \& \ xy \in E(H)\}.$$

Their *lexicographic product*, $G[H]$, has edge set

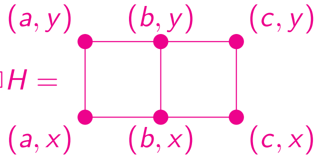
$$E(G[H]) = \{(a,x)(b,y) \mid ab \in E(G), \text{ or } a = b \ \& \ xy \in E(H)\}.$$

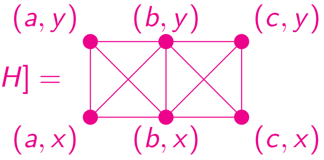
Ex. Consider

$G =$ 

$H =$ 

Then

$G \square H =$ 

$G[H] =$ 

Theorem (HSZ)

Let G be dp with at least two vertices. Then $G[H]$ is dp for any graph H.

Call G *sequentially dp* if its vertex set can be ordered $v_1, v_2, \ldots, v_n$ so that the subgraphs

$$G, G - \{v_1\}, G - \{v_1, v_2\}, \ldots$$

are all isometric in G

Ex. Trees are sequentially dp by the same argument as before.

Clearly G sequentially dp implies G dp. The converse is false.

Theorem (HSZ)

The product $G \Box H$ is sequentially dp if and only if G and H are sequentially dp.

Conjecture (HSZ)

If G and H are dp then so is $G \Box H$.

Note that the converse of this conjecture is false.

THANKS FOR LISTENING!