The cyclic sieving phenomenon - an introduction

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Definitions and an example

Proof by evaluation

Proof by representation theory

A combinatorial proof

Outline

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Definition (Reiner-Stanton-White, 2004)

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4. Three proof techniques: evaluation, representation theory, and combinatorics.

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Let $g \in \mathfrak{S}_n$ *(symmetric group) have disjoint cycle decomposition* $g = g_1 \cdots g_k$. *Let* $T \subseteq [n]$. *Then*

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where $\#g_1 = \ldots = \#g_{n/d} = d$. So, by the second lemma, $T \in \binom{[n]}{k}$ satisfies $gT = T$ iff T is a union of k/d of the n/d cycles g_i. ■

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If G acts on $S = \{s_1, \ldots, s_k\}$ then G also acts on the vector space

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To combinatorially prove $(S, G, f(q))$ exhibits the c.s.p., first find a weight function $\mathrm{wt} : S \to \mathbb{Z}[q]$ such that

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T	: 12	13	14	23	24	34,
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Theorem (Reiner, Stanton, White)

The c.s.p. is exhibited by the triple $\left(\binom{[n]}{k}, C_n, \begin{bmatrix} n \\ k \end{bmatrix}_q \right)$.

Proof (Roichman & S) For $T \in \binom{[n]}{k}$ let $\mathrm{wt}\, T = q^{\sum_{t \in T} t - \binom{k+1}{2}}$.

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Then, by the second lemma, $\#\binom{[n]}{k}^g = \#\binom{[n]}{k}^h$.

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