Pattern avoidance and quasisymmetric functions

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We denote the *n*th symmetric group by

$$\mathfrak{S}_n = \{\sigma \ : \ \sigma \text{ is a permutation of } 1, \ldots, n\}.$$

Given a set of permutation patterns Π we let

$$\mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \ : \ \sigma \text{ avoids every } \pi \in \Pi\}.$$

As a child, Pattern Avoidance liked to compute cardinalities like

$$|\mathfrak{S}_n(\Pi)|.$$

As a teen, Pattern Avoidance took to driving and computing generating functions in one or two variables like

$$\sum_{\sigma \in \mathfrak{S}_n(\Pi)} q^{\operatorname{des} \sigma}.$$

As an adult, Pattern Avoidance started leaping the Tower of London in a single bound and working with generating functions in infinitely many variables.

Let $\mathbf{x} = \{x_1, x_2, \dots\}$. For a monomial in $\mathbf{x}$ we use the notation

$$x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k} = \mathbf{x}_I^N, \quad I = (i_1, i_2, \dots, i_k),\ N = (n_1, n_2, \dots, n_k).$$

Ex. $x_2^7 x_5^9 x_8^3 = \mathbf{x}_{(2,5,8)}^{(7,9,3)}$ which has degree $7 + 9 + 3 = 19$.

The *degree* of x_I^N is defined by $\deg x_I^N = n_1 + n_2 + \dots + n_k$. The set of *formal power series* over the real numbers is

$$\mathbb{R}[[\mathbf{x}]] = \left\{ f(\mathbf{x}) = \sum_{I,N} c_{I,N} \mathbf{x}_I^N \ :\ c_{I,N} \in \mathbb{R} \text{ for all } I, N \right\}.$$

It is an algebra with the usual addition, multiplication, and scalar multiplication of series. Call $f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ *homogeneous of degree n* and write $\deg f(\mathbf{x}) = n$ if we have $\deg \mathbf{x}_I^N = n$ for all monomials $\mathbf{x}_I^N$ in $f(\mathbf{x})$.

Ex. $\deg(x_1^3 x_3^4 + x_1^2 x_2^3 x_4^2) = 7$, but $x_1^2 x_3^4 + x_1^2 x_2^3 x_4^2$ is not homogeneous.

Call $f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ a *symmetric function (SF)* if whenever $\mathbf{x}_I^N$ appears in $f(\mathbf{x})$ and there is a bijection $I \to J$ then the monomial $\mathbf{x}_J^N$ appears in $f(\mathbf{x})$ with the same coefficient.

Ex. $5x_1x_2 + 5x_1x_3 + 5x_2x_3 + \cdots + 7x_1^2x_2 + 7x_1x_2^2 + 7x_1^2x_3 + \ldots$

The set of *symmetric functions homogeneous of degree n* is

$$\mathrm{Sym}_n = \{f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]] \ : \ f(\mathbf{x}) \text{ is a SF and } \deg f(\mathbf{x}) = n\}.$$

This is a vector space over $\mathbb{R}$ with bases indexed by partitions. A weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition of n*, written $\lambda \vdash n$, if we have $\sum_i \lambda_i = n$. The λ_i are called *parts*.

Ex. $\lambda \vdash 4 : \quad (4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)$.

Given $\lambda = (\lambda_1, \dots, \lambda_k)$ the associated *monomial SF* is

$m_\lambda = x_1^{\lambda_1} \dots x_k^{\lambda_k} +$ terms needed to make the function symmetric.

Ex. $m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \dots$

Clearly the m_λ where $\lambda \vdash n$ form a basis for Sym_n.

The *Ferrers diagram* of $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ is an array of left-justified rows of boxes with λ_i boxes in row i. A *standard Young tableau (SYT)* of shape λ is a filling, P, of the Ferrers diagram of λ with $1, \dots, n$ each used exactly once such that rows and columns increase. A *semistandard Young tableau (SSYT)* of shape λ is a filling, T, of the Ferrers diagram of λ with positive integers such that rows weakly increase and columns strictly increase.

Ex. $(3,3,1) =$

, $P =$

1	3	6
2	5	7
4		

, $T =$

1	1	3
2	4	4
6		

$$\mathrm{SYT}(\lambda) := \{P \ : \ P \text{ is a standard Young tableau of shape } \lambda\},$$
$$\mathrm{SSYT}(\lambda) := \{T \ : \ T \text{ is a semistandard Young tableau of shape } \lambda\}.$$

A semistandard Young tableau T has *associated monomial*

$$\mathbf{x}^T = \prod_i x_i^{\text{number of } i\text{'s in } T}.$$

Ex. $T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 6 \\ \hline 2 & 4 & 4 & \\ \cline{1-3}\end{array}$ has $\mathbf{x}^T = x_1^2 x_2 x_3 x_4^2 x_6$.

Another basis of Sym_n uses the *Schur SFs* defined by

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T.$$

Ex. If $\lambda = (2,1)$ then

$$T: \ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \ \ldots, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \ \ldots$$

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \ldots + 2x_1 x_2 x_3 + 2x_1 x_2 x_4 + \ldots$$

Call $f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ a *quasisymmetric function (QSF)* if whenever $\mathbf{x}_I^N$ appears in $f(\mathbf{x})$ and there is a order-preserving bijection $I \to J$ then $\mathbf{x}_J^N$ appears in $f(\mathbf{x})$ with the same coefficient.

Ex. $f(\mathbf{x}) = 6x_1^2x_2 + 6x_1^2x_3 + 6x_2^2x_3 + \dots$

Note that symmetric functions are quasisymmetric, but not conversely. The set of *quasisymmetric functions homogeneous of degree n* is

$$\mathrm{QSym}_n = \{f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]] \ : \ f(\mathbf{x}) \text{ is a QSF and } \deg f(\mathbf{x}) = n\}.$$

This vector space over $\mathbb{R}$ has bases indexed by compositions. A sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a *composition of n*, written $\alpha \models n$, if we have $\sum_i \alpha_i = n$.

Ex. $\alpha \models 3 : \quad (3),\ (2,1),\ (1,2),\ (1,1,1).$

Given $\alpha = (\alpha_1, \ldots, \alpha_k)$ the associated *monomial QSF* is

$M_\alpha = x_1^{\alpha_1} \ldots x_k^{\alpha_k} +$ terms to make the function quasisymmetric.

Ex. $M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \ldots$

Clearly the M_α where $\alpha \models n$ form a basis for QSym_n. Also

$$m_\lambda = \sum_\alpha M_\alpha$$

where the sum is over all rearrangements α of λ.

Ex. $m_{(2,1,1)} = M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)}$

Let $[n] = \{1, 2, \ldots, n\}$. There is a bijection

$$\{\alpha \ : \ \alpha \models n\} \longleftrightarrow \{S \ : \ S \subseteq [n-1]\}$$

by $(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$.

Ex. If $n = 9$ then $(3, 1, 2, 2, 1) \mapsto \{3, 4, 6, 8\}$.

Given $S \subseteq [n-1]$ the associated *fundamental QSF* is

$$F_S = \sum x_{i_1} x_{i_2} \dots x_{i_n}$$

summed over $i_1 \le i_2 \le \dots \le i_n$ with $i_j < i_{j+1}$ if $j \in S$.

Ex. $n = 3$, $S = \{1\}$. Sum over $x_i x_j x_k$ with $i < j \le k$ to get

$$F_{\{1\}} = x_1 x_2^2 + x_1 x_3^2 + \dots + x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots$$

Standard Young tableau P with n elements has *descent set*

$$\text{Des } P = \{i \ : \ i+1 \text{ is in a lower row than } i\} \subseteq [n-1].$$

Theorem (Gessel, 1984)

For any $\lambda \vdash n$

$$s_\lambda = \sum_{P \in \mathrm{SYT}(\lambda)} F_{\text{Des } P}.$$

Ex. Let $\lambda = (3, 2)$.

$$P: \quad \begin{matrix} 1 & 2 & 3 \\ 4 & 5 \end{matrix} \qquad \begin{matrix} 1 & 2 & 4 \\ 3 & 5 \end{matrix} \qquad \begin{matrix} 1 & 2 & 5 \\ 3 & 4 \end{matrix} \qquad \begin{matrix} 1 & 3 & 4 \\ 2 & 5 \end{matrix} \qquad \begin{matrix} 1 & 3 & 5 \\ 2 & 4 \end{matrix}$$

$$s_{(3,2)} = F_{\{3\}} + F_{\{2,4\}} + F_{\{2\}} + F_{\{1,4\}} + F_{\{1,3\}}.$$

At Permutation Patterns 2014, Alex Woo asked the question: is there a way to combine pattern avoidance and quasisymmetric functions? Permutation $\sigma = a_1 a_2 \ldots a_n$ has *descent set* and *descent number*

$$\operatorname{Des}\sigma = \{i \ : \ a_i > a_{i+1}\} \quad \text{and} \quad \operatorname{des}\sigma = |\operatorname{Des}\sigma|.$$

Ex. $\sigma = \overset{1}{5} > \overset{2}{1} \quad \overset{3}{4} \quad \overset{4}{6} > \overset{5}{3} > \overset{6}{2}$, $\operatorname{Des}\sigma = \{1, 4, 5\}$, $\operatorname{des}\sigma = 3$.

Given a set of permutations Π, define

$$Q_n(\Pi) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\operatorname{Des}\sigma}.$$

Questions to ask
(1) When is $Q_n(\Pi)$ symmetric?
(2) If $Q_n(\Pi)$ is symmetric, when does its expansion in the Schur basis have nonnegative coefficients? This is called being *Schur nonnegative*.

Theorem (S)

Suppose $\{123, 321\} \not\subseteq \Pi \subseteq \mathfrak{S}_3$. *TFAE*

1. $Q_n(\Pi)$ *is symmetric for all n.*

2. $Q_n(\Pi)$ *is Schur nonnegative for all n.*

3. Π *is an entry in the following table.*

Π	$Q_n(\Pi)$
$\emptyset$	$\sum_\lambda f^\lambda s_\lambda$
$\{123\}$	$\sum_{c(\lambda)\leq 2} f^\lambda s_\lambda$
$\{321\}$	$\sum_{r(\lambda)\leq 2} f^\lambda s_\lambda$
$\{132, 213\}; \{132, 312\}; \{213, 231\}; \{231, 312\}$	$\sum_{\lambda \text{ a hook}} s_\lambda$
$\{123, 132, 312\}; \{123, 213, 231\}; \{123, 231, 312\}$	$s_{(1^n)} + s_{(2,1^{n-2})}$
$\{132, 213, 321\}; \{132, 312, 321\}; \{213, 231, 321\}$	$s_{(n)} + s_{(n-1,1)}$
$\{132, 213, 231, 312\}$	$s_{(n)} + s_{(1^n)}$.

In all sums λ *runs over partitions of n,* $f^\lambda = |\,\mathrm{SYT}(\lambda)|$, $c(\lambda)$ *and* $r(\lambda)$ *are the number of columns and rows of* λ, *and* 1^k *stands for k copies of the part* 1.

If $\pi = a_1 a_2 \dots a_m$ then $\pi + \ell = (a_1 + \ell)(a_2 + \ell) \dots (a_m + \ell)$.

Ex. If $\pi = 25314$ then $\pi + 2 = 47536$.

If $\pi \in \mathfrak{S}_\ell$ and $\pi' \in \mathfrak{S}_m$ then their *shuffle set* is

$$\pi \sqcup\!\sqcup\, \pi' = \{\sigma \text{ formed from interleaving } \pi \text{ and } \pi' + \ell\}.$$

Ex. $21 \sqcup\!\sqcup\, 12 = \{2134,\ 2314,\ 2341,\ 3214,\ 3241,\ 3421\}$.

Given sets of permutation Π, Π' we let

$$\Pi \sqcup\!\sqcup\, \Pi' = \bigcup_{\pi \in \Pi, \pi' \in \Pi'} \pi \sqcup\!\sqcup\, \pi'.$$

Theorem (Hamaker, Lewis, Pawlowski, S)

For any sets of permutations Π, Π' *and any* n

$$Q_n(\Pi \sqcup\!\sqcup\, \Pi') = Q_n(\Pi') + \sum_{k=0}^{n-1} Q_k(\Pi)(s_1 Q_{n-k-1}(\Pi') - Q_{n-k}(\Pi')).$$

Theorem

$$Q_n(\Pi \sqcup \Pi') = Q_n(\Pi') + \sum_{k=0}^{n-1} Q_k(\Pi)(s_1 Q_{n-k-1}(\Pi') - Q_{n-k}(\Pi')).$$

Corollary (HLPS)

(1) $Q_n(\Pi)$, $Q_n(\Pi')$ are symmetric $\forall n \implies$ so is $Q_n(\Pi \sqcup \Pi')$.
(2) $Q_n(\Pi)$ is Schur nonnegative $\forall n \implies$ so is $Q_n(\Pi \sqcup \mathfrak{S}_m)$ $\forall m$.

Proof.

(1) This follows from the previous theorem and the fact that symmetric functions form an algebra.
(2) Since $\Pi \sqcup \mathfrak{S}_m = \Pi \sqcup \{1\} \sqcup \{1\} \ldots \sqcup \{1\}$, it suffices to prove the result for $\Pi \sqcup \{1\}$. But $\mathfrak{S}_n(1) = \emptyset$ for $n \geq 1$. Thus in the theorem $Q_{n-k}(\Pi') = Q_{n-k}(1) = 0$ and the result follows. □

This corollary explains and generalizes four of results from the first theorem:

$$\{123, 132, 312\} = \{12\} \sqcup \{1\}, \quad \{123, 213, 231\} = \{1\} \sqcup \{12\},$$
$$\{213, 231, 321\} = \{21\} \sqcup \{1\}, \quad \{132, 312, 321\} = \{1\} \sqcup \{21\}.$$

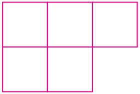
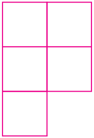
Permutation $\pi = a_1 a_2 \ldots a_n$ has *complement*

$$\pi^c = (n+1-a_1)(n+1-a_2)\ldots(n+1-a_n).$$

Ex. If $\pi = 35421$ then $\pi^c = 31245$.

Clearly $\operatorname{Des}\pi^c = [n-1] \setminus \operatorname{Des}\pi$. We let $\Pi^c = \{\pi^c \ : \ \pi \in \Pi\}$.

The *transpose* of a partition λ is the partition λ^t obtained by reflecting the Ferrers diagram of λ along the main diagonal.

Ex. If $\lambda = (3,2) =$  then $\lambda^t =$  $= (2,2,1)$.

Theorem (HLPS)

(1) $Q_n(\Pi)$ is symmetric if and only if $Q_n(\Pi^c)$ is too. In this case,

$$Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda \iff Q_n(\Pi^c) = \sum_\lambda c_\lambda s_{\lambda^t}.$$

(2) $Q_n(\Pi)$ is Schur nonnegative if and only if $Q_n(\Pi^c)$ is too.

This cuts the work in proving the first theorem by about half.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$, the *$\alpha$-decomposition* of $\pi \in \mathfrak{S}_n$ is

$$\pi = \pi_1\pi_2\ldots\pi_k, \text{ where } |\pi_i| = \alpha_i \text{ for all } i.$$

The *α-descent set* and *α-descent number* of π are

$$\operatorname{Des}_\alpha \pi = \bigcup_i \operatorname{Des} \pi_i \quad \text{and} \quad \operatorname{des}_\alpha \pi = |\operatorname{Des}_\alpha \pi|.$$

Ex. $\pi = 514632, \alpha = (2, 3, 1) \implies \pi = \pi_1\pi_2\pi_3 = 51|463|2.$
So $\pi = 5 > 1 \mid 4\ 6 > 3 \mid 2$ with $\operatorname{Des}_\alpha \pi = \{1, 4\}$ and $\operatorname{des}_\alpha \pi = 2$.

Integer sequence $a_1a_2\ldots a_p$ is *comodal* (complement unimodal) if, for some m,

$$a_1 > a_2 > \cdots > a_m < a_{m+1} < \cdots < a_p.$$

Say $\pi \in \mathfrak{S}_n$ is *α-comodal* if each π_i in its α-decomposition is comodal.

Ex. $\pi = 615438279$ is $(3, 2, 4)$-comodal: $615|43|8279$. It is not $(4, 1, 4)$-comodal: $6154|3|8279$ and 6154 is not comodal.

If Π is a set of permutations then Π_α denotes the α-comodal permutations in Π.

Call $\Pi \subseteq \mathfrak{S}_n$ *fine* if there is an $\mathfrak{S}_n$-character χ with, for all α,

$$\chi(\alpha) = \sum_{\pi \in \Pi_\alpha} (-1)^{\mathrm{des}_\alpha \pi},$$

where $\chi(\alpha)$ is the value of χ on the conjugacy class indexed by α. Examples of fine sets of permutations include
(1) unions of sets of permutations with given inversion number,
(2) unions of conjugacy classes of permutations,
(3) unions of Knuth classes of permutations.
Let

$$\overline{Q}_n(\Pi) = \sum_{\pi \in \Pi} F_{\mathrm{Des}\,\pi}.$$

Theorem (Adin, Roichman)

For $\Pi \subseteq \mathfrak{S}_n$: Π is fine if and only if $\overline{Q}_n(\Pi)$ is Schur nonnegative.

Note that this is a statement about a specific value of n, while the first theorem is a statement for all n.

Other problems to play with.

(1) Define Π and Π' to be *Q-Wilf equivalent* if $Q_n(\Pi) = Q_n(\Pi')$ for all n. What are the Q-Wilf equivalence classes in $\mathfrak{S}_n$?

(2) Stembridge defined an interesting subalgebra of QSym_n call the *peak algebra*. When is $Q_n(\Pi)$ in this subalgebra?

(3) Lam and Pylyavskyy have introduced multi-versions of symmetric functions and of quasisymmetric functions. It would be interesting to study the analogue of $Q_n(\Pi)$ in this context.

Play on!

THANKS FOR LISTENING!