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Restricted growth function patterns and statistics ☆

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A restricted growth function (RGF) of length n is a sequence $w = w_1w_2 \ldots w_n$ of positive integers such that $w_1 = 1$ and $w_i \leq 1 + \max\{w_1, \ldots, w_{i-1}\}$ for $i \geq 2$. RGFs are of interest because they are in natural bijection with set partitions of $\{1, 2, \ldots, n\}$. An RGF w avoids another RGF v if there is no subword of w which standardizes to v. We study the generating functions $\sum_{w\in R_n(v)} q^{\mathrm{st}(w)}$ where $R_n(v)$ is the set of RGFs of length n which avoid v and $\mathrm{st}(w)$ is any of the four fundamental statistics on RGFs defined by Wachs and White. These generating functions exhibit interesting connections with multiset permutations, integer partitions,

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Noncrossing partition
Nonnesting partition
Pattern
rb
rs
Restricted growth function
Partition
Statistic
Two-colored Motzkin path

and two-colored Motzkin paths, as well as noncrossing and nonnesting set partitions.

1. Introduction

Recently, there has been a flurry of activity looking at the distribution of statistics over pattern classes in various objects. For example, see [4,5,7,9,11,12,14]. There are two notions of pattern containment for set partitions, one obtained by standardizing a subpartition and one obtained by standardizing a subword of the corresponding restricted growth function. In [6], the present authors studied the distribution of four fundamental statistics of Wachs and White [25] over avoidance classes using the first definition. The purpose of this paper is to carry out an analogous investigation for the second.

Let us begin by defining our terms. Consider a finite set S. A *set partition* σ of S is a family of nonempty subsets $B_1, \ldots, B_k$ whose disjoint union is S, written $\sigma = B_1/\ldots/B_k \vdash S$. The B_i are called *blocks* and we will usually suppress the set braces and commas in each block for readability. We will be particularly interested in set partitions of $[n] := \{1, 2, \ldots, n\}$ and will use the notation

$$\Pi_n = \{\sigma \ : \ \sigma \vdash [n]\}.$$

To illustrate $\sigma = 145/2/3 \vdash [5]$. If $T \subseteq S$ and $\sigma = B_1/\ldots/B_k \vdash S$ then there is a corresponding *subpartition* $\sigma' \vdash T$ whose blocks are the nonempty intersections $B_i \cap T$. To continue our example, if $T = \{2, 4, 5\}$ then we get the subpartition $\sigma' = 2/45 \vdash T$.

The concept of pattern is built on the standardization map. Let O be an object with labels which are positive integers. The *standardization* of O, $\mathrm{stan}(O)$, is obtained by replacing all occurrences of the smallest label in O by 1, all occurrences of the next smallest by 2, and so on. Say that $\sigma \vdash [n]$ *contains* π *as a pattern* if it contains a subpartition σ' such that $\mathrm{stan}(\sigma') = \pi$. In this case σ' is called an *occurrence* or *copy* of π in σ. Otherwise, we say that σ *avoids* π and let

$$\Pi_n(\pi) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi\}.$$

In our running example, $\sigma = 145/2/3$ contains $\pi = 1/23$ since $\mathrm{stan}(2/45) = 1/23$. But σ avoids $12/3$ because if one takes any two elements from the first block of σ then it is impossible to find an element from another block bigger than both of them. Klazar [16–18] was the first to study this approach to set partition patterns. For more recent work, see the paper of Bloom and Saracino [3].

To define the second notion of pattern containment for set partitions, we need to introduce restricted growth functions. A sequence $w = w_1 w_2 \ldots w_n$ of positive integers is a *restricted growth function* (RGF) if it satisfies the conditions

1. $w_1 = 1$, and
2. for $i \geq 2$ we have

$$w_i \leq 1 + \max\{w_1, \ldots, w_{i-1}\}. \tag{1}$$

For example, $w = 11213224$ is an RGF, but $w = 11214322$ is not since $4 > 1 + \max\{1, 1, 2, 1\}$. The number of elements of w is called its *length* and denoted $|w|$. Define

$$R_n = \{w \ : \ w \text{ is an RGF of length } n\}.$$

For RGFs and even more general sequences we will use w_i as the notation for the ith element of w.

To connect RGFs with set partitions, we will henceforth write all $\sigma = B_1/B_2/\ldots/B_k \vdash [n]$ in *standard form* which means that

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

Note that this implies $\min B_1 = 1$. Given $\sigma = B_1/\ldots/B_k \vdash [n]$ in standard form, we construct an associated word $w(\sigma) = w_1 \ldots w_n$ where

$$w_i = j \text{ if and only if } i \in B_j.$$

More generally, for any set P of set partitions, we let $w(P)$ denote the set of $w(\sigma)$ for $\sigma \in P$. Returning to our running example, we have $w(145/2/3) = 12311$. It is easy to see that σ being in standard form implies $w(\sigma)$ is an RGF and that the map $\sigma \mapsto w(\sigma)$ is a bijection $\Pi_n \to R_n$.

We can now define patterns in terms of RGFs. Given RGFs v, w we call v a *pattern* in w if there is a subword w' of w with $\mathrm{stan}(w') = v$. The use of the terms "occurrence," "copy," and "avoids" in this setting are the same as for set partitions. Given v we define the corresponding *avoidance class*

$$R_n(v) = \{w \in R_n \ : \ w \text{ avoids } v\}.$$

Similarly define, for any set V of RGFs,

$$R_n(V) = \{w \in R_n \ : \ w \text{ avoids every } v \in V\}.$$

As before, consider $w = w(145/2/3) = 12311$. Then w contains $v = 121$ because either of the subwords 121 or 131 of w standardize to v. However, w avoids $v = 122$ since the only repeated elements of w are ones. Note that this is in contrast to the fact that $145/2/3$ contains $1/23$ where $w(1/23) = 122$. In general, we have the following result whose proof is straightforward and left to the reader.

Proposition 1.1. *Suppose that partitions π and σ have RGFs $v = w(\pi)$ and $w = w(\sigma)$. If w contains v then σ contains π, but not necessarily conversely. Equivalently, we have $R_n(v) \supseteq w(\Pi_n(\pi))$.* □

Sagan [20] described the sets $R_n(v)$ for all $v \in R_3$. To state this result, we need a few more definitions. The *initial run* of an RGF w is the longest prefix of the form $12\ldots m$. Write a^l to indicate a string of l copies of the integer a. Call the word w *layered* if it has the form $w = 1^{n_1}2^{n_2}\ldots m^{n_m}$, equivalently, if it is weakly increasing. The next theorem will be useful in the sequel.

Theorem 1.2 *([20]). We have the following characterizations.*

1. $R_n(111) = \{w \in R_n$: *every element of w appears at most twice*$\}$.
2. $R_n(112) = \{w \in R_n$: *w has initial run $12\ldots m$ and $m \geq w_{m+1} \geq w_{m+2} \geq \cdots \geq w_n$*$\}$.
3. $R_n(121) = \{w \in R_n$: *w is layered*$\}$.
4. $R_n(122) = \{w \in R_n$: *every element $j \geq 2$ of w appears only once*$\}$.
5. $R_n(123) = \{w \in R_n$: *w contains only 1s and 2s*$\}$. □

Using this result, it is not hard to compute the cardinalities of the classes.

Corollary 1.3 *([20]). We have*

$$\#R_n(112) = \#R_n(121) = \#R_n(122) = \#R_n(123) = 2^{n-1}$$

and

$$\#R_n(111) = \sum_{i\geq 0} \binom{n}{2i}(2i)!!$$

where $(2i)!! = 1 \cdot 3 \cdot 5 \cdot \ \ldots \ \cdot (2i-1)$. □

Our object, in part, is to prove generalizations of the formulae in this corollary using the statistics of Wachs and White and their generating functions. They defined four statistics on RGFs denoted lb, ls, rb, and rs where the letters l, r, b, and s stand for left, right, bigger, and smaller, respectively. We will explicitly define the lb statistic and the others are defined analogously. Given a word $w = w_1w_2\ldots w_n$, let

$$\mathrm{lb}(w_j) = \#\{w_i \ : \ i < j \text{ and } w_i > w_j\}.$$

Otherwise put, $\mathrm{lb}(w_j)$ counts the number of integers which are to the left of w_j in w and bigger than w_j. Note that multiple copies of the same integer left of and bigger than w_j are only counted once. Note, also, that $\mathrm{lb}(w_j)$ depends on w and not just the value of w_j. But context will ensure that there is no confusion. For an example, if $w = 12332412$ then for $w_5 = 2$ we have $\mathrm{lb}(w_5) = 1$ since three is the only larger integer which occurs before the two. For w itself, define

$$\mathrm{lb}(w) = \mathrm{lb}(w_1) + \mathrm{lb}(w_2) + \cdots + \mathrm{lb}(w_n).$$

Continuing our example,

$$\mathrm{lb}(12332412) = 0 + 0 + 0 + 0 + 1 + 0 + 3 + 2 = 6.$$

For an RGF v, consider the generating function

$$\mathrm{LB}_n(v) = \mathrm{LB}_n(v; q) = \sum_{w \in R_n(v)} q^{\mathrm{lb}(w)}$$

and similarly for the other three statistics. Sometimes we will be able to prove things about multivariate generating functions such as

$$F_n(v) = F_n(v; q, r, s, t) = \sum_{w \in R_n(v)} q^{\mathrm{lb}(v)} r^{\mathrm{ls}(v)} s^{\mathrm{rb}(v)} t^{\mathrm{rs}(v)}.$$

As noted in Proposition 1.1, if $v = w(\pi)$ then we always have $R_n(v) \supseteq w(\Pi_n(\pi))$. But for certain π we have equality. In particular, as shown in [20], this is true for $\pi = 123, 13/2, 1/2/3$ and the corresponding $v = 111, 121, 123$. So these patterns will not, for the most part, be analyzed in what follows since their generating functions were computed in [6].

The rest of this paper is organized as follows. In the next section, we consider the generating functions for the two remaining patterns of length three, namely $v = 112$ and 122. Interestingly, multiset permutations, integer partitions, and q-binomial coefficients come into play as well as connections with the polynomials for the remaining patterns of length three. In Section 3, we consider the classes $R_n(V)$ for all $V \subseteq R_3$ containing two or more patterns. The next three sections deal with RGFs of length longer than three. Section 4 gives recursive methods for calculating generating functions for longer patterns in terms of shorter ones. The following two sections are concerned with the patterns 1212 and 1221 which are connected with noncrossing and nonnesting set partitions, respectively. These results are closely related to two-colored Motzkin paths. We note that Simion [22] previously considered single and joint distributions of the Wachs and White statistics over RGFs for noncrossing partitions, and we rederive one of her results using a different proof. We end with a section of comments and open questions.

2. Single patterns of length 3

2.1. Patterns related to multiset permutations

In this subsection, we show bijectively that three of the generating functions under consideration are the same. Moreover, the common value can be expressed in terms of the q-binomial coefficients which count multiset permutations. First we need some definitions.

We let $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. We can now define a q-analogue of the factorial, letting $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Finally, we define the *q-binomial coefficients* or *Gaussian polynomials* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

By convention, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = 0$ if $k < 0$ or $k > n$.

Now that we have the proper notation, we can state our first equidistribution theorem.

Theorem 2.1. *We have*

$$\mathrm{LB}_n(112) = \mathrm{RS}_n(112) = \mathrm{LB}_n(122) = \sum_{m \geq 1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q.$$

We will establish the statements for avoiding 112 first and then deal with 122. First, we need to recall a well-known combinatorial interpretation for the Gaussian polynomials. The *inversion number* of any sequence $w = w_1 \dots w_n$ of integers is

$$\mathrm{inv}(w) = \#\{(i,j) \mid i < j,\ w_i > w_j\}.$$

Let $P_{n,k}$ denote the set of all permutations $\pi = \pi_1 \dots \pi_n$ of a multiset consisting of $n-k$ ones and k twos. Then

$$\sum_{\pi \in P_{n,k}} q^{\mathrm{inv}(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2}$$

In the next proof we use the notation $w + k$ for the result of adding the integer k to every element of the sequence w.

Proposition 2.2. *We have*

$$\mathrm{LB}_n(112) = \sum_{m \geq 1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q = \mathrm{RS}_n(112).$$

Proof. Let

$$R_{n,m}(112) = \{w \in R_n(112) \mid \max w = m\}$$

and throughout the proof m will denote the maximum of w. To prove the first equality it suffices, by equation (2), to find a bijection $\alpha : R_{n,m}(112) \to P_{n-1,m-1}$ such that $\mathrm{lb}(w) = \mathrm{inv}(\alpha(w))$. Every RGF either ends with a one or with an element greater than one. From Theorem 1.2 it follows that the $w \in R_n(112)$ ending with a one are precisely those of the form $w = v1$ where $v \in R_{n-1}(112)$. Similarly, those ending with a greater number are those which can be written $w = 1(v+1)$. Note that $\mathrm{lb}(v1) = m - 1 + \mathrm{lb}(v)$ and $\mathrm{lb}(1(v+1)) = \mathrm{lb}(v)$. Now define α recursively by $\alpha(1) = \emptyset$ and for $n \geq 2$

$$\alpha(w) = \begin{cases} \alpha(v)1 & \text{if } w = v1, \\ \alpha(v)2 & \text{if } w = 1(v+1). \end{cases}$$

It is easy to prove inductively that α is a bijection. And it is also not hard to see that α sends lb to inv. For example, if $w = v1$ then by induction

$$\mathrm{lb}(w) = \mathrm{lb}(v1) = m - 1 + \mathrm{lb}(v) = m - 1 + \mathrm{inv}(\alpha(v)) = \mathrm{inv}(\alpha(v)1) = \mathrm{inv}(\alpha(w))$$

with the other case being similar.

To prove the second equality, we must find a bijection $\beta : R_{n,m}(112) \to P_{n-1,m-1}$ such that $\mathrm{rs}(w) = \mathrm{inv}(\beta(w))$. The construction is similar, except that in the case $w = v1$ we make the more refined decomposition $w = 1^k(v+1)1^\ell$ where $k, \ell \geq 1$ and $v \in R_{n-k-\ell}(112)$. The definition of β is then $\beta(1^n) = 1^{n-1}$ and for $w \neq 1^n$

$$\beta(w) = \begin{cases} 1^{k+\ell-2}2\beta(v)1 & \text{if } w = 1^k(v+1)1^\ell, \\ \beta(v)2 & \text{if } w = 1(v+1). \end{cases}$$

The reader should have no difficulties filling in the rest of the proof. □

Proposition 2.3. *We have*

$$\mathrm{LB}_n(112) = \mathrm{LB}_n(122).$$

Proof. We will construct a bijection $\eta : R_n(112) \to R_n(122)$ such that $\mathrm{lb}(w) = \mathrm{lb}(\eta(w))$. Let $w \in R_n(112)$ have maximum m. To construct $\eta(w)$ we start with the sequence $12\ldots m$. For every w_i, where w_i is not in the initial run of w, we will insert a 1 just to the right of element $m - w_i + 1$ in $\eta(w)$. Note that $1 \leq w_i \leq m$ ensures that this element always exists, and in conjunction with Theorem 1.2 this shows that η is well-defined. For example if $w = 12345664331$ then $\eta(w) = 11231411561$. Clearly η is invertible.

To check that lb is preserved, note that in w the initial run does not contribute to lb and in $\eta(w)$, none of the terms greater than 1 contribute to lb. Consider w_i such that

$i > m$. Then $\mathrm{lb}(w_i) = m - w_i$. If we examine the 1 placed into $\eta(w)$ because of w_i, we notice that it has $m - w_i$ terms greater than 1 to its left. Therefore the lb of this 1 is $m - w_i$. Thus, $\mathrm{lb}(w) = \mathrm{lb}(\eta(w))$. □

Combining the above propositions yields Theorem 2.1.

2.2. Patterns related to integer partitions with distinct parts

Next, we will explore a connection to integer partitions with distinct parts. An *integer partition* is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$. The λ_i are called *parts* and we say the parts are *distinct* if the sequence is strictly decreasing. It is well-known that the generating function for partitions with distinct parts of size at most $n - 1$ is

$$\prod_{i=1}^{n-1}(1 + q^i).$$

As noted in the introduction, for the pattern 121 we have $R_n(121) = w(\Pi_n(13/2))$. So we can use the following result of Goyt and Sagan who studied the ls statistic on $\Pi_n(13/2)$.

Proposition 2.4 *([12]). We have*

$$\mathrm{LS}_n(121) = \mathrm{RB}_n(121) = \prod_{i=1}^{n-1}(1 + q^i).$$

The following result establishes that, once again, four of our generating functions are the same.

Theorem 2.5. *We have the equalities*

$$\mathrm{LS}_n(112) = \mathrm{LS}_n(121) = \mathrm{RB}_n(121) = \mathrm{RB}_n(122) = \prod_{i=1}^{n-1}(1 + q^i).$$

As before, we break the proof of this result into pieces.

Proposition 2.6. *We have*

$$\mathrm{LS}_n(112) = \mathrm{LS}_n(121).$$

Proof. We will construct a bijection $\xi : R_n(112) \to R_n(121)$ such that $\mathrm{ls}(w) = \mathrm{ls}(\xi(w))$. Given $w \in R_n(112)$ we will construct $\xi(w)$ by rearranging the elements of w in weakly increasing order. By Theorem 1.2, this is well defined. For the inverse, if we are given a layered RGF, v, then we use the first element of each layer to form an initial run and rearrange the remaining elements in weakly decreasing order.

For any RGF $w = w_1 \ldots w_n$ we have $\mathrm{ls}(w_i) = w_i - 1$. Since w and $\xi(w)$ are rearrangements of each other, ls is preserved. $\square$

Proposition 2.7. *We have*

$$\mathrm{LS}_n(112) = \mathrm{RB}_n(122).$$

Proof. Let $\eta : R_n(112) \to R_n(122)$ be as in Proposition 2.3. To see that $\mathrm{ls}(w) = \mathrm{rb}(\eta(w))$, first note that $\mathrm{ls}(w_i) = w_i - 1$. By construction, the initial run of w has ls that is equal to the total rb of the first occurrences of elements in $\eta(w)$. In addition, for each w_i not in the initial run of w, we place a 1 to the right of $m - w_i + 1$ in $\eta(w)$, and therefore there are $w_i - 1$ elements to its right that are larger than it. Thus $\mathrm{ls}(w) = \mathrm{rb}(\eta(w))$. $\square$

Combining the above propositions, we obtain Theorem 2.5.

2.3. Patterns not related to integer partitions

In this section, we present two more connections between the generating functions of patterns of length 3. The first is as follows.

Theorem 2.8. *We have*

$$\mathrm{RS}_n(122) = \mathrm{LB}_n(123) = \mathrm{RS}_n(123) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k.$$

Proof. It was shown in [6] that

$$\mathrm{LB}_n(123) = \mathrm{RS}_n(123) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k.$$

So it suffices to construct a bijection $f : R_n(122) \to R_n(123)$ which preserves the rs statistic. First, recall that by Theorem 1.2, words in $R_n(123)$ contain only 1s and 2s and that for $w \in R_n(122)$, every element $j \geq 2$ of w appears only once. Given $w = w_1 \ldots w_n \in R_n(122)$, we will construct $f(w) = u_1 \ldots u_n$ by replacing each element $j \geq 2$ in w with a 2. This is a bijection, as any word in $R_n(122)$ is uniquely determined by the placement of its ones. In addition, $\mathrm{rs}(w_i) = \mathrm{rs}(u_i)$ by construction so that $\mathrm{rs}(w) = \mathrm{rs}(f(w))$. $\square$

The second establishes yet another connection between statistics on $R_n(112)$ and $R_n(122)$.

Theorem 2.9. *We have*

$$\mathrm{RB}_n(112) = \mathrm{LS}_n(122) = \sum_{m=0}^{n} \binom{n-1}{n-m} q^{\binom{m}{2}}.$$

Proof. For the first equality, let $\eta : R_n(112) \to R_n(122)$ be as in Propositions 2.3 and 2.7. We will show that for $w \in R_n(112)$, we have $\mathrm{rb}(w) = \mathrm{ls}(\eta(w))$. Because w is unimodal, only the initial run contributes to rb. If m is the largest element in the initial run of w, then $\mathrm{rb}(w) = 1 + 2 + \cdots + (m-1) = \binom{m}{2}$. Similarly, only the elements greater than 1 in $\eta(w)$ contribute to ls. By construction, the largest element in $\eta(w)$ is m as well. Thus, $\mathrm{ls}(\eta(w)) = 1 + 2 + \cdots + (m-1) = \binom{m}{2}$.

To show that $\mathrm{RB}_n(112) = \sum_m \binom{n-1}{n-m} q^{\binom{m}{2}}$ it suffices, as can be seen from the previous paragraph, to count the number of $w \in R_n(112)$ with initial run $12\ldots m$. Notice that once the elements in the weakly decreasing sequence following the initial run have been selected, there is only one way to order them. For that sequence we must choose $n-m$ elements from the set $[m]$, allowing repetition, yielding a total of $\binom{n-1}{n-m}$ as desired. $\square$

It is remarkable that the map η connects so many of the statistics on $R_n(112)$ and $R_n(122)$; see the proofs of Propositions 2.3, 2.7, and Theorem 2.9. The four-variable generating functions $F_n(v; q, r, s, t)$ can be used to succinctly summarize these demonstrations as follows.

Theorem 2.10. *We have*

$$F_n(112; q, r, s, 1) = F_n(122; q, s, r, 1). \quad \square$$

3. Multiple patterns of length 3

This section considers RGFs which avoid multiple patterns of length three. In all cases we are able to determine the four-variate generating function. We find connections to Gaussian polynomials, integer partitions, and Fibonacci numbers.

For sets $V \subseteq R_3$ it is not hard to see if $121 \in V$ or $\{111, 122\} \subseteq V$ then the two notions of pattern avoidance, as RGFs and as set partitions, are equivalent. In these cases the characterization and cardinality of $R_n(V)$ have been determined by Goyt [10], and the generating functions $F_n(V)$ have been determined by [6]. For completeness we will include the characterization, cardinality, and generating function for all $V \subseteq R_3$ except those which contain both 111 and 123 since in these cases $R_n(V) = \emptyset$ for $n \geq 5$.

The following characterizations are obtained directly by taking the intersection of the sets described in Theorem 1.2 so the proof is omitted.

Proposition 3.1. *We have the following characterizations for $n \geq 1$.*

1. $R_n(111, 112) = \{w \in R_n \ : \ w$ *has initial run* $12 \dots m$ *and* $m \geq w_{m+1} > w_{m+2} > \cdots > w_n\}$.
2. $R_n(111, 121) = \{w \in R_n \ : \ w$ *is layered and every element of* w *appears at most twice*$\}$.
3. $R_n(111, 122) = \{w \in R_n \ : \ w = 12 \dots n$ *or* $w = 12 \dots i1(i+1) \dots (n-1)$ *for some* $0 < i < n\}$.
4. $R_n(112, 121) = \{w \in R_n \ : \ w = 12 \dots mm \dots m$ *for some* $1 \leq m \leq n\}$.
5. $R_n(112, 122) = \{w \in R_n \ : \ w = 12 \dots m11 \dots 1$ *for some* $1 \leq m \leq n\}$.
6. $R_n(112, 123) = \{w \in R_n \ : \ w = 12^i 1^{n-i-1}$ *for some* $0 \leq i < n\}$.
7. $R_n(121, 122) = \{w \in R_n \ : \ w = 1^i 23 \dots (n-i+1)$ *for some* $0 < i \leq n\}$.
8. $R_n(121, 123) = \{w \in R_n \ : \ w = 1^i 2^{n-i}$ *for some* $0 < i \leq n\}$.
9. $R_n(122, 123) = \{w \in R_n \ : \ w = 1^n$ *or* $w = 1^i 2 1^{n-i-1}$ *for some* $0 < i < n\}$.

Let f_n denote the nth Fibonacci number defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Corollary 3.2. *The cardinality of the avoidance sets above are as follows.*

1. $\#R_n(111, 112) = \#R_n(111, 121) = f_n$.
2. *For all other pairs of length 3 patterns* $\{v_1, v_2\}$ *except for the pair* $\{111, 123\}$ *we have* $\#R_n(v_1, v_2) = n$.

Proof. The sizes for $\{111, 121\}$, $\{111, 122\}$, $\{112, 121\}$, $\{121, 122\}$, and $\{121, 123\}$ were computed by Goyt [10]. The rest follow easily from Proposition 3.1 except $\#R_n(111, 112)$.

We will now show that $\#R_n(111, 112)$ satisfies the Fibonacci recurrence. It is not hard to see that $\#R_0(111, 112) = \#R_1(111, 112) = 1$. Next consider $w \in R_n(111, 112)$ with $n \geq 2$. We know from Proposition 3.1 that $w = 12 \dots m a_{m+1} \dots a_n$ where $m \geq a_{m+1} > \cdots > a_n$ which implies that w has either one 1 or two 1's. If w has one 1 then that 1 is at the beginning and $w = 1(v+1)$ for some $v \in R_{n-1}(111, 112)$. If w has two 1's then the second 1 will be a_n and $w = 1(v+1)1$ for some $v \in R_{n-2}(111, 112)$. This gives us the desired recurrence $\#R_n(111, 112) = \#R_{n-1}(111, 112) + \#R_{n-2}(111, 112)$. $\square$

All the sets described in Proposition 3.1 are sufficiently simple that we can determine their four-variable generating functions. Many of the functions can be simplified by extending the Gaussian polynomials which were defined at the beginning of Section 2 to two variables. The bivariate analogue of n is

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + q^{n-1}$$

so $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \dots [1]_{p,q}$ and the binomial analogue is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

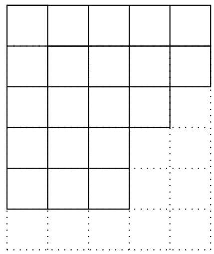

Fig. 1. The Young diagram for $\lambda = (5, 5, 4, 3, 3)$ in the 6×5 rectangle β.

We can recover the one-variable version by letting $p = 1$.

For our next result, we will need a classic interpretation for the q-binomial coefficients in terms of integer partitions. Before we give the interpretation we need some definitions. The *Young diagram* of a partition $\lambda = (\lambda_1, \dots \lambda_k)$ is an array of left-justified rows of boxes with row i containing λ_i boxes. The Young diagram of $\lambda = (5, 5, 4, 3, 3)$ is given in Fig. 1. Given β which is an $r \times \ell$ rectangle of boxes and a partition $\lambda = (\lambda_1, \dots \lambda_k) \subseteq \beta$ as Young diagrams, we define its *complement* λ^c to be the partition which is composed of all boxes in β outside of λ rotated 180°. Continuing our example from Fig. 1, the partition $\lambda = (5, 5, 4, 3, 3)$ in the 6×5 rectangle has $\lambda^c = (5, 2, 2, 1)$ as its compliment. Let $|\lambda| = \sum_i \lambda_i$ and note that $|\lambda^c| = r\ell - |\lambda|$. For β, an $r \times \ell$ rectangle, we have the well-known formula

$$\begin{bmatrix} r+\ell \\ \ell \end{bmatrix}_{p,q} = \sum_{\lambda \subseteq \beta} p^{|\lambda|} q^{|\lambda^c|}.$$

Almost all the functions $F_n(V)$ for $V = \{v_1, v_2\} \subset R_3$ are computed in [6], [12], or follow easily from methods previously used in this paper, so the proofs will be omitted. The only exception is $V = \{111, 112\}$ for which we will provide a demonstration. The method we use parallels the proof used for $F_n(111, 121)$ in [12].

Theorem 3.3. *We have the following generating functions.*

1. $F_n(111, 112) = \sum_{m \geq 0} (qrt^2)^{\binom{m}{2}} (rs)^{\binom{n-m}{2}} \begin{bmatrix} n-m \\ m \end{bmatrix}_{qt,r}$.
2. $F_n(111, 121) = \sum_{m \geq 0} (rs)^{\binom{m}{2} + \binom{n-m}{2}} \begin{bmatrix} n-m \\ m \end{bmatrix}_{r,s}$. *See [12, page 244].*
3. $F_n(111, 122) = (rs)^{\binom{n}{2}} + (rs)^{\binom{n-1}{2}} [n-1]_{qt,s}$. *See [6, Theorem 7.1].*
4. $F_n(112, 121) = \sum_{m=1}^{n} r^{(m-1)(n-m)} (rs)^{\binom{m}{2}}$. *See [6, Theorem 7.1].*
5. $F_n(112, 122) = (rs)^{\binom{n}{2}} + \sum_{m=1}^{n-1} q^{(m-1)(n-m)} (rs)^{\binom{m}{2}} t^{m-1}$.
6. $F_n(112, 123) = 1 + r^{n-1}s + qrst[n-2]_{q,rt}$.

7. $F_n(121,122) = \displaystyle\sum_{m=1}^{n} (rs)^{\binom{m}{2}} s^{(m-1)(n-m)}$. *See [6, Theorem 7.1].*
8. $F_n(121,123) = 1 + rs[n-1]_{r,s}$. *See [6, Theorem 7.1].*
9. $F_n(122,123) = 1 + rs^{n-1} + qrst[n-2]_{q,s}$.

Proof. Define the set of pairs

$$P_n = \bigcup_{m\geq 1} \{(\lambda,\beta) : \lambda \text{ is a partition with distinct parts, } \beta = (n-m)\times(m-1), \text{ and } \lambda \subseteq \beta\}.$$

Let $w \in R_n(111,112)$ so that, by Proposition 3.1, $w = 12\ldots m w_{m+1}\ldots w_n$ where $m = \max w$ and the sequence is strictly decreasing from w_{m+1} on. Now define $\rho : R_n(111,112) \to P_n$ by $\rho(w) = (\lambda,\beta)$ where

$$\lambda = (m - w_n, m - w_{n-1}, \ldots, m - w_{m+1})$$

and β is as in the definition of P_n. Note that we are permitting the last part of λ to be zero. It is not hard to see that ρ is well defined and invertible.

If $w \mapsto (\lambda,\beta)$ then we claim

$$(\mathrm{lb}(w), \mathrm{ls}(w), \mathrm{rb}(w), \mathrm{rs}(w)) = \left(|\lambda|,\ \binom{m}{2} + |\lambda^c|,\ \binom{m}{2},\ \binom{n-m}{2} + |\lambda|\right).$$

Indeed, $\mathrm{lb}(w) = |\lambda|$ since only the w_i for $i > m$ contribute to lb and in that case $\mathrm{lb}(w_i) = m - w_i$ which is a part of λ. For $\mathrm{ls}(w)$, the binomial coefficient comes from the initial run, while for $i > m$ we have

$$\mathrm{ls}(w_i) = w_i - 1 = (m-1) - (m - w_i) = \lambda^c_{i-m}.$$

Only the initial run contributes to $\mathrm{rb}(w)$, giving $\binom{m}{2}$. Call the subword $w_{m+1}w_{m+1}\ldots w_n$ of w its *tail*. Since the tail is strictly decreasing, it will contribute $\binom{n-m}{2}$ to $\mathrm{rs}(w)$. If w_i is in the initial run, then $\mathrm{rs}(w_i)$ is the number of elements in the tail smaller than w_i. But this is the same as the number of boxes in column $m-i+1$ of λ and so the initial run adds another $|\lambda|$ to $\mathrm{rb}(w)$.

There is a standard bijection δ from partitions with r distinct parts $\lambda = (\lambda_1,\ldots,\lambda_r)$ to ordinary partitions $\mu = (\mu_1,\ldots,\mu_r)$ with r parts where in both case we permit zero as a part. It is given by

$$\delta(\lambda_1,\lambda_2,\ldots,\lambda_{r-1},\lambda_r) = (\lambda_1 - (r-1), \lambda_2 - (r-2), \ldots, \lambda_{r-1} - 1, \lambda_r) = \mu.$$

Note that $|\lambda| = |\mu| + \binom{r}{2}$. Also, if $\lambda \subseteq r\times\ell$ then $\mu \subseteq r \times (\ell - r + 1)$. Furthermore $\mu^c = \delta(\lambda^c)$.

Now if $\lambda \subseteq (n-m)\times(m-1)$ then $\delta(\lambda) \subseteq (n-m)\times(2m-n)$. It follows that we have a bijection $\rho' : R_n(111,112) \to P'_n$ where

$$P'_n = \bigcup_{m\geq 1} \{(\mu,\gamma) : \mu \text{ a partition, } \gamma = (n-m)\times(2m-n), \text{ and } \mu \subseteq \gamma\}.$$

Furthermore, if $\rho'(w) = (\mu,\gamma)$ then

$$\begin{aligned}&(\mathrm{lb}(w), \mathrm{ls}(w), \mathrm{rb}(w), \mathrm{rs}(w))\\ &\quad = \left(\binom{n-m}{2} + |\mu|,\ \binom{m}{2} + \binom{n-m}{2} + |\mu^c|,\ \binom{m}{2},\ 2\binom{n-m}{2} + |\mu|\right).\end{aligned}$$

Translating this bijection into a generating function identity and then replacing m by $n-m$ yields the desired equation. □

From the functions provided in Theorem 3.3 we can see several symmetries and invariants.

Corollary 3.4. *Let $V \subseteq R_3$ and $n \geq 0$.*

1. *For $V = \{v_1, v_2\}$ such that $121 \in V$ or $V = \{111, 122\}$, the function $F_n(V)$ is invariant under switching q and t.*
2. *The following sets V have $F_n(V)$ invariant under switching r and s.*

$$\{111,121\}, \{112,122\}, \{121,123\}.$$

3. *We have the equalities*

$$\begin{aligned} F_n(111,121; q,r,s,t) &= F_n(111,112; s,r,s,1),\\ F_n(112,121; q,r,s,t) &= F_n(121,122; q,s,r,t), \end{aligned}$$

and

$$F_n(112,123; q,r,s,1) = F_n(122,123; q,s,r,1). \quad \square$$

The avoidance classes for $V \subseteq R_3$ of size three and four are easy to determine and so we will merely list them in Table 1. The reader interested in the corresponding generating functions will be able to easily write them down.

4. Recursive formulae and longer words

In this section we will investigate generating functions for avoidance classes of various RGFs of length greater than three. This includes a recursive formula for computing the generating functions for longer words in terms of shorter ones.

Table 1
Avoidance classes for $V \subset R_3$ of size three and four and $n \geq 3$.

V	$R_n(V)$
$\{111, 112, 121\}$	$12\ldots n,\ 12\ldots(n-2)(n-1)^2$
$\{111, 112, 122\}$	$12\ldots n,\ 12\ldots(n-1)1$
$\{111, 121, 122\}$	$12\ldots n,\ 1123\ldots(n-1)$
$\{112, 121, 122\}$	$12\ldots n,\ 1^n$
$\{112, 121, 123\}$	$1^n,\ 12^{n-1}$
$\{112, 122, 123\}$	$1^n,\ 121^{n-2}$
$\{121, 122, 123\}$	$1^n,\ 1^{n-1}2$
$\{111, 112, 121, 122\}$	$12\ldots n$
$\{112, 121, 122, 123\}$	1^n

Recall that $w + k$ denotes the word obtained by adding the nonnegative integer k to every element of w. Note that if w is an RGF and k is nonzero, then $w + k$ will not be an RGF. However, the word $\bar{w} = 12\ldots k(w + k)$ obtained by concatenating the increasing sequence $12\ldots k$ with $w + k$, will be an RGF. In fact, there is a relationship between the generating functions for w and $\bar{w}$. In the following theorem, we show that this relationship holds for the ls and rs statistics. We note that in [19, Propositions 2.1 and 2.2], Mansour and Shattuck use the same method to find the cardinalities of the avoidance classes of the pairs of patterns $\{1222, 12323\}$ and $\{1222, 12332\}$.

Theorem 4.1. *Let v be an RGF and $\bar{v} = 1(v + 1)$. Then*

$$\mathrm{LS}_n(\bar{v}) = \sum_{j=0}^{n-1} \binom{n-1}{j} q^j \,\mathrm{LS}_j(v)$$

and

$$\mathrm{RS}_n(\bar{v}) = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{n+k-j-2}{k} q^k \,\mathrm{RS}_j(v).$$

Proof. We start by building the avoidance class of $\bar{v}$ out of the avoidance class of v. We do so by taking a word w in the avoidance class of v, forming $1(w+1)$, and then adding a sufficient number of ones to $1(w + 1)$ to obtain a word $\bar{w}$ of length n which avoids $\bar{v}$. We then count how adding these ones affects the respective statistics.

We first establish that avoidance is preserved in this process. Let $w \in R_j(v)$. Since w avoids v, we know $1(w + 1)$ avoids $1(v + 1) = \bar{v}$. Now we need to show that forming $\bar{w}$ by adding $n - j - 1$ ones to $1(w + 1)$ in any manner will result in $\bar{w}$ avoiding $\bar{v}$. If $\bar{w} \notin R_n(\bar{v})$, then there is a subword w' of w such that $\mathrm{stan}(w') = \bar{v}$. Since $\bar{v} = 1(v + 1)$, the smallest element of w' must appear only at the beginning of the subword, and must be a 1 since $1(w+1)$ avoided $\bar{v}$. But removing the unique 1 and standardizing the remaining elements shows that there is a subword of w that standardizes to v. This is a contradiction. Therefore, we must have $\bar{w} \in R_n(\bar{v})$. Similarly, every word in $R_n(\bar{v})$ with $n-j$ ones can be turned into a word in $R_j(v)$ by removing all ones and standardizing. If this word isn't in $R_j(v)$, then it contains a subword that standardized to v. As before, this means the original word contains $1(v+1) = \bar{v}$, which is a contradiction. Therefore, we can construct every word in $R_n(\bar{v})$ from the words in $R_j(v)$ for $j \in [0, n-1]$.

We now translate this process into the generating function identities. First we will focus on the LS formula. We can choose any $w \in R_j(v)$, and place the elements of $w+1$ in our word $\bar{w}$ in $\binom{n-1}{j}$ different ways since we must leave the first position free to be a one. Then we fill in the rest of the positions with ones. Since we added 1 to each element of $w \in R_j(v)$ and added a one to the beginning of the word, we have $\mathrm{ls}(\bar{w}) = \mathrm{ls}(w) + j$. So

$$\mathrm{LS}_n(\bar{v}) = \sum_{\bar{w} \in R_n(\bar{v})} q^{\mathrm{ls}(\bar{w})} = \sum_{j=0}^{n-1} \sum_{w \in R_j(v)} \binom{n-1}{j} q^j q^{\mathrm{ls}(w)} = \sum_{j=0}^{n-1} \binom{n-1}{j} q^j \, \mathrm{LS}_j(v).$$

For the RS formula, instead of all j elements of $w+1$ increasing the statistic, only the k elements of $w+1$ that are to the left of the rightmost one in $\bar{w}$ will contribute. If we choose where to place these elements, then everything else is forced. We start with $n-1$ positions available, and disregard $j-k+1$ for the rightmost one and the elements of $w+1$ that appear after it. Thus we have $(n-1)-(j-k+1) = n+k-j-2$ positions to choose from. Summing over all values of j and k gives the RS formula. □

In the paper of Dokos et al. [7], the authors introduced the notion of statistical Wilf equivalence. We will consider how this idea can be applied to the four statistics we have been studying. We define two RGFs v and w to be ls-*Wilf-equivalent* if $\mathrm{LS}_n(v) = \mathrm{LS}_n(w)$ for all n, and denote this by

$$v \overset{\mathrm{ls}}{\equiv} w.$$

Similarly define an equivalence relation for the other three statistics. Let st denote any of our four statistics. Given any equivalence $v \overset{\mathrm{st}}{\equiv} w$, we can generate an infinite number of related equivalences.

Corollary 4.2. *Suppose $v \overset{\mathrm{st}}{\equiv} w$. Then for any $k \geq 1$ we have*

$$12\ldots k(v+k) \overset{\mathrm{st}}{\equiv} 12\ldots k(w+k).$$

Proof. For $\mathrm{st} = \mathrm{ls}, \mathrm{rs}$ this follows immediately from Theorem 4.1 and induction on k. For the other two statistics, note that the same ideas as in the proof of Theorem 4.1 can be used to show that one can write down the generating function for st over

$R_n(12\dots k(v+k))$ in terms of the generating functions for st over $R_j(v)$ for $j \le n$ although the expressions are more complicated. Thus induction can also be used in these cases as well. $\square$

Applying this corollary to the equivalences in Theorem 2.3, Proposition 2.6, and Theorem 2.8 yields the following result.

Corollary 4.3. *We have*

$$12\dots kk(k+1) \overset{\text{lb}}{\equiv} 12\dots k(k+1)(k+1),$$
$$12\dots kk(k+1) \overset{\text{ls}}{\equiv} 12\dots k(k+1)k,$$
$$12\dots k(k+1)(k+1) \overset{\text{rs}}{\equiv} 12\dots k(k+1)(k+2),$$

for all $k \ge 1$. $\square$

We will now demonstrate how these ideas can be used to find the generating functions for a family of RGFs by finding $\mathrm{LS}_n(12\dots k)$ for a general k. We begin by finding the degree of $\mathrm{LS}_n(12\dots k)$ through a purely combinatorial approach before using Theorem 4.1 to give a formula for the generating function itself.

Proposition 4.4. *For* $n \ge k$, *the generating function* $\mathrm{LS}_n(12\dots k)$ *is monic and*

$$\deg \mathrm{LS}_n(12\dots k) = \binom{k-2}{2} + (k-2)(n-k+2).$$

Proof. It is easy to see that $w \in R_n(12\dots k)$ if and only if $w_i < k$ for all i. Also $\mathrm{ls}(w_i) = w_i - 1$ for all i. Thus there is a unique word maximizing ls, namely $w = 12\dots(k-2)(k-1)\dots(k-1)$. Thus $\mathrm{LS}_n(12\dots k)$ is monic with $\mathrm{ls}(w) = 0+1+2+\dots+(k-3)+(n-k+2)(k-2) = \binom{k-2}{2} + (k-2)(n-k+2)$. $\square$

To obtain a formula for $L_n(12\dots k)$ we will use the q-analogues introduced earlier, often suppressing the subscript q for readability. Consider the rational function of q

$$K_{m,n} = \frac{[m+1]^{n-1} - 1}{[m]}.$$

We will need the following facts about $K_{m,n}$. Writing $[m+1]^{n-1} = (1+q[m])^{n-1}$ and expanding by the binomial theorem gives

$$K_{m,n} = \sum_{j=1}^{n-1} \binom{n-1}{j} q^j [m]^{j-1}. \tag{3}$$

We also have

$$\frac{1}{[m]}(K_{m+1,n} - K_{1,n}) = \sum_{j=1}^{n-1} \binom{n-1}{j} q^j K_{m,j} \tag{4}$$

which can be obtained by substituting the definition of $K_{m,j}$ into the sum and then applying the previous equation.

Finally we define, for $k \geq 3$,

$$c_k = 1 - \sum_{j=1}^{k-3} \frac{1}{[j]!} c_{k-j}.$$

Note that when $k = 3$ the sum is empty and so $c_3 = 1$. Note also that for fixed k, the number of terms in $LS_n(12\ldots k)$ is a linear function of n by Proposition 4.4. However, in the formula for this generating function which we give next the number of summands only depends on k, making it an efficient way to compute this polynomial.

Theorem 4.5. *For $k \geq 3$, we have*

$$\mathrm{LS}_n(12\ldots k) = 1 + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} K_{i,n}.$$

Proof. We proceed with a proof by induction. In [6], the authors show that $\mathrm{LS}_n(1/2/3) = [2]^{n-1}$ for the set partition $1/2/3$. Recall that a set partition avoids $1/2/3$ if and only if its corresponding RGF avoids 123. Therefore $\mathrm{LS}_n(1/2/3) = \mathrm{LS}_n(123) = [2]^{n-1}$ for $n \geq 1$. Rewriting this as $\mathrm{LS}_n(123) = 1 + K_{1,n}$ gives our base case for $k = 3$.

Suppose the equality held for $k \geq 3$. Then, using Theorem 4.1 as well as equations (3) and (4),

$$\begin{aligned}
\mathrm{LS}_n(12\ldots k+1) &= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j \,\mathrm{LS}_j(12\ldots k) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j \left(1 + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} K_{i,j}\right) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} q^j + \sum_{i=1}^{k-2} \frac{1}{[i-1]!} c_{k-i+1} \left(\sum_{j=1}^{n-1} \binom{n-1}{j} q^j K_{i,j}\right) \\
&= 1 + K_{1,n} + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} (K_{i+1,n} - K_{1,n}) \\
&= 1 + K_{1,n} \left(1 - \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1}\right) + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} K_{i+1,n}
\end{aligned}$$

$$= 1 + c_{k+1}K_{1,n} + \sum_{i=1}^{k-2} \frac{1}{[i]!} c_{k-i+1} K_{i+1,n}$$
$$= 1 + \sum_{i=1}^{k-1} \frac{1}{[i-1]!} c_{k-i+2} K_{i,n}$$

which completes the induction. □

Let 1^m denote the RGF consisting of m copies of one. The ideas in the proof of Theorem 4.1 can be used to give recursive formulae for this pattern. It would be interesting to find other patterns where this reasoning could be applied.

Theorem 4.6. *For $m \geq 0$, we have*

$$LS_n(1^m) = \sum_{j=1}^{m-1} \binom{n-1}{j-1} q^{n-j}\, \mathrm{LS}_{n-j}(1^m)$$

and

$$\mathrm{RS}_n(1^m) = \mathrm{RS}_{n-1}(1^m) + \sum_{j=2}^{m-1} \sum_{k=0}^{n-j} \binom{j+k-2}{k} q^k\, \mathrm{RS}_{n-j}(1^m).$$

Proof. Let w avoid 1^m. Then w can be uniquely obtained by taking a w' avoiding 1^m and inserting j ones in $w'+1$, where $1 \leq j \leq m-1$ and a one must be inserted at the beginning of the word. The formula for $\mathrm{LS}_n(1^m)$ now follows since the binomial coefficient counts the number of choices for the non-initial ones, $\mathrm{LS}_{n-j}(1^m)$ is the contribution from $w'+1$, and q^{n-j} is the obtained from the interaction between the initial one and $w'+1$. The reader should now have no problem modifying the proof of the $\mathrm{RS}_n(\bar{v})$ formula in Theorem 4.1 to apply to this case. □

5. The pattern 1212

5.1. Noncrossing partitions

The set partitions which avoid the pattern 13/24 are called *non-crossing* and are of interest, in part, because of their connection with Coxeter groups and the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

See the memoir of Armstrong [1] for more information. In this case the set containment in Proposition 1.1 can be turned into an equality. Note that $w(13/24) = 1212$. For the next three results, we will only sketch the proofs since the reader can easily supply the details.

Proposition 5.1. *We have*

$$R_n(1212) = w(\Pi_n(13/24)).$$

Proof. As just noted, it suffices to show that if π contains 13/24, then $w(\pi)$ contains 1212. Now $w(\pi)$ contains a subword $xyxy$ for some $x \neq y$ and it is easy to check that, by the RGF condition, the first such occurrence has $x < y$ as desired. □

We now focus on gaining information about these partitions by studying $R_n(1212)$. We begin by applying the rs statistic to $R_n(1212)$, and in doing so obtain a q-analogue of the standard Catalan recursion. We first need the following lemma regarding 1212-avoiding restricted growth functions.

Lemma 5.2. *For an RGF w, the following are equivalent:*

(1) *The RGF w avoids* 1212*.*
(2) *There are no $xyxy$ subwords in w.*
(3) *If $w_i = w_{i'}$ for some $i < i'$ then, for all $j' > i'$, either $w_{j'} \leq w_{i'}$ or $w_{j'} > \max\{w_1, \ldots, w_{i'}\}$.*

Proof. The equivalence of the first two statements follows from the proof of Proposition 5.1. The proof that (2) implies (3) is by contradiction. If there is a j' with $j' > i'$ and $w_{i'} < w_{j'} \leq \max\{w_1, \ldots w_{i'}\}$ then there must be $j < i'$ with $w_j = w_{j'}$. The two cases $j < i$ and $j > i$ both lead to a contradictory copy of $xyxy$. The reverse implication can also be proved by contradiction using the method of proof in Proposition 5.1. □

We now move to a recursive way of producing words in $R_n(1212)$.

Corollary 5.3. *If u is in $R_{n-1}(1212)$ then both $1u$ and $1(u+1)$ are in $R_n(1212)$.*

Proof. By the previous lemma, we know that u does not contain any $xyxy$ subwords. It is now easy to check that in this case neither does $1u$ or $1(u+1)$ so that, again by the previous lemma, both avoid 1212. □

With these results in hand, we move to one of the main results of this section. For two words w and u, we will use the set notation $w \cap u = \emptyset$ to denote that w and u have no elements in common. The next theorem gives a q-analogue of the usual recursion for the Catalan numbers. It will also be used to establish a connection between $R_n(1212)$ and lattice paths.

Theorem 5.4. *We have*

$$\mathrm{RS}_0(1212) = 1,$$
$$\mathrm{RS}_1(1212) = 1,$$

and for $n \geq 2$,

$$\mathrm{RS}_n(1212) = 2\,\mathrm{RS}_{n-1}(1212) + \sum_{k=1}^{n-2} q^k\,\mathrm{RS}_k(1212)\,\mathrm{RS}_{n-k-1}(1212).$$

Proof. The base cases are trivial. To prove the recursion, we partition $R_n(1212)$ into three disjoint subsets X, Y, and Z as follows:

$$X = \{w \in R_n(1212)\ :\ w_1 = 1 \text{ and there are no other 1s in } w\},$$
$$Y = \{w \in R_n(1212)\ :\ w_1w_2 = 11\},$$
$$Z = \{w \in R_n(1212)\ :\ w_1w_2 = 12 \text{ and there is at least one other 1 in } w\}.$$

We claim that we can also describe X as the set of words defined by

$$X = \{w = 1(u+1)\ :\ u \in R_{n-1}(1212)\}. \tag{5}$$

To see this, let u be a word in $R_{n-1}(1212)$. From Corollary 5.3, we know $w = 1(u+1)$ is an element of $R_n(1212)$, and by definition of $u+1$, the only 1 in w will be w_1. This gives one containment. Now let w be an element of X as originally defined. Since the leading one in w is unique, let $u+1$ denote the last $n-1$ letters in w. By Lemma 5.2, w contains no $xyxy$ subword; in particular, $u+1$ contains no $xyxy$ subword. Standardizing $u+1$ to the RGF u will not create any $xyxy$ subwords, and thus u will be contained in $R_{n-1}(1212)$. This gives the reverse containment, from which we conclude that the two sets are equal. A similar proof, without standardization of the subword, allows us to describe Y as the set

$$Y = \{w = 1u\ :\ u \in R_{n-1}(1212)\}. \tag{6}$$

Now note that for any RGF u, we have $\mathrm{rs}(u) = \mathrm{rs}(1(u+1))$ and $\mathrm{rs}(u) = \mathrm{rs}(1u)$. Using this fact, and the above characterization of the sets, we can see that X and Y must contribute $\mathrm{RS}_{n-1}(1212)$ each to the total $\mathrm{RS}_n(1212)$ polynomial.

Finally, we claim that we can characterize Z as

$$\begin{aligned} Z = \{w = 1(u+1)1v\ :\ & u \in R_k(1212) \text{ for } 1 \leq k \leq n-2, \\ & \mathrm{stan}(1v) \in R_{n-k-1}(1212), v \cap (u+1) = \emptyset\}. \end{aligned} \tag{7}$$

First, let w be contained in Z as defined at the beginning of the proof. By definition of Z, w has a nonempty subword of the form $u+1$ consisting of all entries between the first and second 1 in w. Let the length of u be k. As with the set X, $u+1$ will standardize to u, an RGF in $R_k(1212)$. Now let v be the last $n-k-2$ letters in w, so that our word is of the form

$$w = 1(u+1)1v.$$

Since a 1 is repeated before v, we must have $v_i = 1$ or $v_i > \max(u+1)$ for all i by Lemma 5.2, where v_i is the ith letter of v. This gives $v \cap (u+1) = \emptyset$. Furthermore, there is no $xyxy$ subword contained in $1v$, and standardizing the subword will not create an $xyxy$ pattern. Thus $\operatorname{stan}(1v)$ is contained in $R_{n-k-1}(1212)$. This shows one inclusion between the two versions of Z. Now let u be an element of $R_k(1212)$, and let $1v'$ be an element of $R_{n-k-1}(1212)$. Corollary 5.3 gives that $1(u+1)$ avoids 1212 as well. Now from the RGF $1v'$, we create the word $1v$ by setting

$$(1v)_i = \begin{cases} (1v')_i & \text{if } (1v')_i = 1 \\ (1v')_i + \max(u) & \text{if } (1v')_i \neq 1. \end{cases}$$

We claim that $w = 1(u+1)1v$ is a member of $R_n(1212)$. To see this, note that $u+1$ contains no $xyxy$ subwords, and further $u+1$ shares no integers in common with the rest of w. Therefore $u+1$ cannot contribute to an $xyxy$ subword in w. Thus if such a subword exists in w, it must also exist in $11v$. This is impossible as it implies an $xyxy$ subword in $1v'$, contradicting our choice of $1v'$. We have now shown the reverse set containment, which implies the desired equality of the two sets.

With this characterization of Z, we can now decompose $\operatorname{rs}(w)$ for w in Z as

$$\operatorname{rs}(w) = \operatorname{rs}(u+1) + k + \operatorname{rs}(1v),$$

where the middle term comes from the contribution to rs caused by comparing the elements of $u+1$ with the second 1 in w. Summing over all possibilities of k, u, and v, and noting that the rs of a word is not affected by standardization, we can see that Z will contribute

$$\sum_{k=1}^{n-2} q^k \operatorname{RS}_k(1212) \operatorname{RS}_{n-k-1}(1212).$$

Adding the results obtained from X, Y, and Z now gives the desired total. $\square$

For the next result, we first recall the definition of a Motzkin path. A Motzkin path P of length n is a lattice path in the plane which starts at $(0,0)$, ends at $(n,0)$, stays weakly above the x-axis, and which uses vector steps in the form of up steps $[1,1]$,

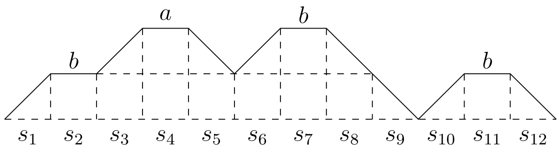

Fig. 2. A two-colored Motzkin path.

horizontal steps $[1, 0]$, and down steps $[1, -1]$. Let $\mathcal{M}_n$ denote the set of all Motzkin paths of length n. We write $P = s_1 \dots s_n$ for such a path, where

$$s_i = \begin{cases} U \text{ if the } i\text{th step is an up step,} \\ H \text{ if the } i\text{th step is a horizontal step,} \\ D \text{ if the } i\text{th step is a down step.} \end{cases}$$

Given a step s_i in P, we can realize s_i geometrically as a line segment in the plane connecting two lattice points in the obvious way. Fig. 2 displays the Motzkin path $P = UHUHDUHDDUHD$. Define the *level* of s_i, $l(s_i)$, to be the lowest y-coordinate in s_i. Continuing with our example path, the sequence of levels of the steps is $0, 1, 1, 2, 1, 1, 2, 1, 0, 0, 1, 0$. Note that the level statistic provides a natural pairing of up steps with down steps in a Motzkin path. Namely, we associate an up step s_i with the first down step s_j, $j > i$, which is at the same level as s_i, i.e. $l(s_i) = l(s_j)$. We will call such steps *paired*. In Fig. 2 the pairs are s_1 and s_9, s_3 and s_5, s_6 and s_8, and s_{10} and s_{12}.

We now define a *two-colored Motzkin path* P of length n to be a Motzkin path of length n whose horizontal steps are individually colored using one of the colors a or b. We will call an a-colored horizontal step an *a-step* and a b-colored horizontal step a *b-step*. For a two-colored Motzkin path $P = s_1 \dots s_n$ we will still use s_i equal to U or D for up steps and down steps, but will use a or b instead of H to show the color of the horizontal steps. In this notation, our example path is $P = UbUaDUbDDUbD$. Let $\mathcal{M}_n^2$ denote the set of all two-colored Motzkin paths of length n. For two paths $P = s_1 \dots s_n$ and $Q = t_1 \dots t_m$ we write $PQ = s_1 \dots s_n t_1 \dots t_m$ to indicate their concatenation. Interestingly, Wachs and White were originally inspired to look at the RGF statistics because of a question posed by Dennis Stanton (personal communication) and motivated by the appearance two-colored Motzkin paths in a combinatorial interpretation of the moments of q-Charlier polynomials given by Viennot [24].

Let the *area* of a path P, $\text{area}(P)$, denote the area enclosed between P and the x-axis. Our example has $\text{area}(P) = 14$. Defining

$$M_n(q) = \sum_{P \in \mathcal{M}_n^2} q^{\text{area}(P)}, \tag{8}$$

Drake [8] proved the following recursion.

Theorem 5.5 *([8]). We have $M_0(q) = 1$ and, for all $n \geq 1$,*

$$M_n(q) = 2M_{n-1}(q) + \sum_{k=1}^{n-2} q^k M_k(q) M_{n-k-1}(q). \quad \square$$

Using Theorems 5.4 and 5.5 as well as induction on n immediately gives the following equality.

Corollary 5.6. *We have, for all $n \geq 1$,*

$$\mathrm{RS}_n(1212) = M_{n-1}(q). \quad \square$$

Interestingly, it turns out that we also have $\mathrm{LB}_n(1212) = \mathrm{LB}_n(1221) = M_{n-1}(q)$ which will be proved in Section 6. In our next result, we prove the previous corollary directly via a bijection between $\mathcal{M}^2_{n-1}$ and $R_n(1212)$. Although various bijections can be obtained by composing those already in the literature on Catalan combinatorics, we have not been able to construct the one we need in this manner.

Theorem 5.7. *There is an explicit bijection $\psi : R_n(1212) \to \mathcal{M}^2_{n-1}$ such that $\mathrm{rs}(w) = \mathrm{area}(\psi(w))$ for all $w \in R_n(1212)$.*

Proof. We define the map inductively, maintaining the partition $R_n(1212) = X \cup Y \cup Z$ developed in the proof of Theorem 5.4. For $n = 1$ we simply map $w = 1$ to the empty Motzkin path. For $n > 1$ we define

$$\psi(w) = \begin{cases} b\psi(u) & \text{if } w \in X \text{ and } w = 1(u+1), \\ a\psi(u) & \text{if } w \in Y \text{ and } w = 1u, \\ U\psi(u)D\psi(\mathrm{stan}(1v)) & \text{if } w \in Z \text{ and } w = 1(u+1)1v. \end{cases}$$

It is straightforward to show that ψ is bijective by induction. One can also inductively show that $\mathrm{rs}(w) = \mathrm{area}(\psi(w))$. Indeed, if $w \in R_n(1212)$ and $w \in X$ with $w = 1(u+1)$, then

$$\mathrm{rs}(w) = \mathrm{rs}(u+1) = \mathrm{rs}(u) = \mathrm{area}(\psi(u)) = \mathrm{area}(b\psi(u)) = \mathrm{area}(\psi(w)).$$

The cases $w \in Y$ and $w \in Z$ follow similarly. $\square$

One can also give an explicit, non-recursive, formula for the map ψ in the preceding proof. Specifically, if $w = w_1 \ldots w_n \in R_n(1212)$, then we have $\psi_n(w) = s_1 \ldots s_{n-1}$, where

$$s_i = \begin{cases} a \text{ if } w_i = w_{i+1}, \\ b \text{ if } w_i < w_{i+1} \text{ and there does not exist } j > i+1 \text{ with } w_i = w_j, \\ U \text{ if } w_i < w_{i+1} \text{ and there exists } j > i+1 \text{ with } w_i = w_j, \\ D \text{ if } w_i > w_{i+1}. \end{cases}$$

The equivalence of the two descriptions of ψ can be shown by induction. Similarly, one can derive an explicit formula for ψ^{-1}. If $P = s_1 \dots s_{n-1} \in \mathcal{M}^2_{n-1}$, then we have $\psi_n^{-1}(P) = w_1 \dots w_n$, with $w_1 = 1$ and

$$w_{i+1} = \begin{cases} 1 + \max\{w_1, \dots, w_i\} & \text{if } s_i \text{ equals } U \text{ or } b, \\ w_i & \text{if } s_i = a, \\ w_j & \text{if } s_i = D \text{ is paired with the up step } s_j. \end{cases}$$

5.2. Combinations with other patterns

Next we examine RGFs that avoid 1212 and another pattern of length 3. As the patterns 121, 122, and 112 are all subpatterns of 1212, the only interesting cases to look at are $R_n(111, 1212)$ and $R_n(123, 1212)$. We start by calculating $\mathrm{RS}_n(111, 1212)$. It is easy to combine Theorem 1.2 and Lemma 5.2 to characterize $R_n(111, 1212)$.

Lemma 5.8. *We have, for all $n \geq 0$,*

$$R_n(111, 1212) = \{w \in R_n(1212) \ : \ \textit{every element of } w \textit{ appears at most twice}\}. \qquad \square$$

The following proposition is similar to Theorem 5.4 in many respects. First, this proposition provides a q-analogue of the standard Motzkin recursion and is proved using techniques similar to those used previously. Furthermore, it will also be used to connect $R_n(111, 1212)$ to lattice paths.

Proposition 5.9. *We have*

$$\mathrm{RS}_0(111, 1212) = 1,$$
$$\mathrm{RS}_1(111, 1212) = 1,$$

and for $n \geq 2$,

$$\mathrm{RS}_n(111, 1212) = \mathrm{RS}_{n-1}(111, 1212) + \sum_{k=0}^{n-2} q^k \,\mathrm{RS}_k(111, 1212)\,\mathrm{RS}_{n-k-2}(111, 1212).$$

Proof. We follow the proof of Theorem 5.4 by partitioning $R_n(111, 1212)$ into the sets

$$X = \{w \in R_n(111, 1212) \ : \ w_1 = 1 \text{ and there are no other 1s in } w\},$$

$$\begin{aligned} Y &= \{w \in R_n(111, 1212) \ : \ w_1 w_2 = 11\}, \\ Z &= \{w \in R_n(111, 1212) \ : \ w_1 w_2 = 12 \text{ and there is a single other 1 in } w\}. \end{aligned}$$

Using the same reasoning as in Theorem 5.4 and adding the restrictions of avoiding 111 gives the equivalent characterizations

$$\begin{aligned} X &= \{w = 1(u+1) \ : \ u \in R_{n-1}(111, 1212)\}, \\ Y &= \{w = 11(u+1) \ : \ u \in R_{n-2}(111, 1212)\}, \\ Z &= \{w = 1(u+1)1v \ : \ u \in R_k(111, 1212) \text{ for } 1 \le k \le n-2, \\ &\qquad \operatorname{stan}(v) \in R_{n-k-2}(111, 1212), v \cap 1(u+1) = \emptyset\}. \end{aligned}$$

From this, the desired recurrence easily follows. □

The next result provides an explicit bijection between $R_n(111, 1212)$ and $\mathcal{M}_n$. We first extend the level statistic defined in the previous subsection to paths. Given a Motzkin path $P = s_1 \ldots s_n$, we define the level of the path, $l(P)$, to be

$$l(P) = \sum_{i=1}^{n} l(s_i).$$

In Fig. 2, $l(P) = 10$. It should be noted that if we impose a rectangular grid of unit squares on the first quadrant of the plane, then $l(P)$ simply counts the total area of the unit squares contained below P and above the x-axis. We will use our bijection to calculate the generating function for the level statistic taken over all Motzkin paths of length n.

Theorem 5.10. *For $n \ge 0$, we have*

$$\mathrm{RS}_n(111, 1212) = \sum_{P \in \mathcal{M}_n} q^{l(P)}.$$

Proof. As with Theorem 5.7 we inductively build a bijection $\phi : R_n(111, 1212) \to \mathcal{M}_n$ so that $\mathrm{rs}(w) = l(\phi(w))$ for each $w \in R_n(111, 1212)$ and for each $n \ge 1$. For $n = 1$ we map $w = 1$ to the single step Motzkin path H. For $n > 1$ we write $R_n(111, 1212) = X \cup Y \cup Z$ as in the proof of Proposition 5.9 and define

$$\phi(w) = \begin{cases} H\phi(u) & \text{if } w \in X \text{ with } w = 1(u+1), \\ UD\phi(u) & \text{if } w \in Y \text{ with } w = 11(u+1), \\ U\phi(u)D\phi(\operatorname{stan}(v)) & \text{if } w \in Z \text{ with } w = 1(u+1)1v. \end{cases}$$

As with the map in Theorem 5.7, ϕ is seen to be bijective via a simple induction. That $\mathrm{rs}(w) = l(\phi(w))$ also follows inductively. If $w \in R_n(111, 1212)$ and $w \in X$ with $w = 1(u+1)$, then

$$\mathrm{rs}(w) = \mathrm{rs}(u+1) = \mathrm{rs}(u) = l(\phi(u)) = l(H\phi(u)) = l(\phi(w)).$$

The equality similarly holds for $w \in Y$ or $w \in Z$, and thus the result follows. $\square$

As with ψ, one can also derive an explicit formula for ϕ. If $w \in R_n(111, 1212)$ with $w = w_1 \dots w_n$, then $\phi(w) = s_1 \dots s_n$ with

$$s_i = \begin{cases} U \text{ if } w_i = w_j \text{ for some } j > i, \\ H \text{ if } w_i \neq w_j \text{ for all } j \neq i, \\ D \text{ if } w_i = w_j \text{ for some } j < i. \end{cases}$$

For the inverse map, let $P = s_1 \dots s_n \in \mathcal{M}_n$. Then $\phi^{-1}(P) = w_1 \dots w_n$, where $w_1 = 1$ and, for $i \geq 2$,

$$w_i = \begin{cases} 1 + \max\{w_1, \dots, w_{i-1}\} & \text{if } s_i = U \text{ or } H, \\ w_j & \text{if } s_i \text{ is a down step paired with } s_j. \end{cases}$$

We conclude this section with a simple proposition characterizing $R_n(123, 1212)$. As the result follows easily from Theorem 1.2, Proposition 5.2, and standard counting techniques, we leave the proof to the reader.

Proposition 5.11. *If w is contained in $R_n(123, 1212)$, then*

$$w = 1^l 2^i 1^{n-i-l}$$

for some $l \geq 1$, $i \geq 0$ satisfying $l + i \leq n$. As such, for $n \geq 0$ we have

$$\mathrm{LB}_n(123, 1212) = \mathrm{RS}_n(123, 1212) = 1 + \sum_{k=0}^{n-2} (n-k-1)q^k$$

and

$$\mathrm{LS}_n(123, 1212) = \mathrm{RB}_n(123, 1212) = 1 + \sum_{k=1}^{n-1} (n-k)q^k. \quad \square$$

6. The pattern 1221

6.1. Nonnesting partitions

The term “nonnesting” has been defined in different ways in the literature. In some sources a nonnesting partition is a partition π where we can never find four elements

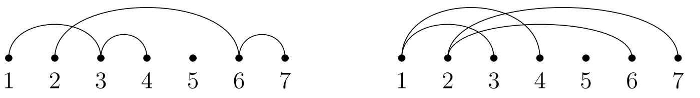

Fig. 3. The arc diagram and left arc diagram for the partition 134/267/5.

$a < x < y < b$ such that $a, b \in A$ and $x, y \in B$ for two distinct blocks A, B. This is the sense used in Klazar's paper [16] and is equivalent to a partition avoiding 14/23.

In other papers, including Klazar's article [18], a partition π is nonnesting if, whenever there are four elements $a < x < y < b$ such that $a, b \in A$ and $x, y \in B$ for two distinct blocks A, B, then there exists a $c \in A$ such that $x < c < y$. This definition is often given using arc diagrams. We draw the *arc diagram* of a partition of $[n]$ by writing 1 through n on a straight line and drawing arcs (a, b) if $a < b$ are in a block and consecutive when writing the block in increasing order, see Fig. 3. A *nesting* is a pair of arcs (a, b) and (x, y) such that $a < x < y < b$, and we will say in this case that the pair of arcs *nest*. It is not hard to see that having no nesting arcs is equivalent to the second definition of a nonnesting partition. And it is known that the number of partitions satisfying either of these two equivalent conditions is the Catalan number, C_n.

There is another notion of nonnesting which we will call left nonnesting and can be defined by a different collection of arcs. For each block B we will draw all arcs of the form $(\min B, b)$ with $b \in B \backslash \{\min B\}$, and call the diagram with these arcs the *left arc diagram*. An example is displayed in Fig. 3. Left arc diagrams were also defined and studied by Kim [15] who called them front representations of set partitions. If a partition's left arc diagram has no pair of arcs which nest then we will call this partition *left nonnesting* to distinguish our term from the previous two definitions of nonnesting. Let this set be LNN_n.

Proposition 6.1. *We have*

$$R_n(1221) = w(\mathrm{LNN}_n).$$

Proof. We will just show that if a partition's left arc diagram contains a nesting then its associated RGF has the pattern 1221 since the proof of the reverse inclusion is similar. Let $\pi = B_1/\ldots/B_k$ be a partition of $[n]$. Say that its left arc diagram has a nesting which means that we have arcs (a, b) and (x, y) such that $a < x < y < b$. Since these are arcs from the left arc diagram we know that $a = \min B_i$ and $x = \min B_j$ for some distinct blocks B_i and B_j, and since we order the blocks of π so that their minimum elements increase we know that $i < j$. As result $w(\pi)$ has the subword $ijji$ which is the pattern 1221. $\square$

In [13], Jelínek and Mansour defined nonnesting by requiring that a partition's associated RGF avoid 1221, and they showed that $|R_n(1221)| = C_n$, the nth Catalan number. From the previous proposition, it follows that LNN_n defines the same set of partitions and so $|\,\mathrm{LNN}_n\,| = C_n$.

The rest of this section will describe $R_n(1221)$, some of its generating functions, and some connections to other patterns. We will prove that $\mathrm{LB}_n(1221) = \mathrm{RS}_n(1212)$ by showing that there exists a bijection from two-colored Motzkin paths to $R_n(1221)$ which maps area to lb, and then the result will follow from Theorem 5.7. We further use this bijection and previous methods to determine the generating function for RGFs that avoid some pair of patterns which include 1221. We end the section by showing $\mathrm{LB}_n(1221) = \mathrm{LB}_n(1212)$ and summarizing all the equalities we have proved.

6.2. *The pattern* 1221 *by itself*

For an RGF $w = w_1 \dots w_n$ we will call a letter w_i *repeated* if there exists a $j < i$ such that $w_j = w_i$. If a letter is not a repeated letter, we will call it a *first occurrence*. Since w is an RGF, the first occurrences are exactly the *left to right maxima*, that is, the elements w_i such that $w_i > \max\{w_1, \dots, w_{i-1}\}$.

Lemma 6.2. *A word $w \in R_n(1221)$ if and only if the subword of all repeated letters in w is weakly increasing.*

Proof. Say that w contains the pattern 1221 and so has a subword $xyyx$ for some $x < y$. The second yx are repeated letters in w. This implies that there is a decrease in the subword of all repeated letters.

Conversely, say that the subword of all repeated letters of w has an decrease yx with $x < y$. Since these are repeated letters in an RGF the first y of w appears earlier, and the first x in w appears earlier than the first y. Hence we have a subword $xyyx$ with $x < y$ and the pattern 1221. $\square$

Using the previous lemma we can define a surjection $\mathrm{inc} : R_n \to R_n(1221)$. The map will take a $w \in R_n$ and will output $\mathrm{inc}(w) = v$ which is w with its subword of repeated letters put in weakly increasing order. For example if $w = 1112221331$ then $\mathrm{inc}(w) = 1112112323$.

To see this map is well defined we must first show that v is an RGF. But the subword of repeated letters is rearranged to be weakly increasing which forces the maximum of any prefix to weakly decrease. Since the left to right maxima of w do not move in this process, they do not change in passing to v so that the latter is still an RGF. Also, v avoids 1221 by Lemma 6.2, showing inc is well defined.

In the next lemma we show that inc preserves lb. Note that because w is an RGF, all the numbers in the interval $[w_i + 1, \max\{w_1, \dots, w_{i-1}\}]$ appear to the left of w_i and are larger than w_i, so

$$\mathrm{lb}(w_i) = \max\{w_1, \dots, w_{i-1}\} - w_i. \tag{9}$$

Lemma 6.3. *Let v be a rearrangement of w such that both have the same left to right maxima in the same places. Then* $\mathrm{lb}(v) = \mathrm{lb}(w)$. *In particular,* $\mathrm{lb}(w) = \mathrm{lb}(\mathrm{inc}(w))$.

Proof. Since w and v only have their repeated letters rearranged and their left to right maxima fixed, we know $\max\{w_1, \ldots, w_i\} = \max\{v_1, \ldots, v_i\}$ for all i and $\{v_1, \ldots, v_n\} = \{w_1, \ldots, w_n\}$ as multisets. Using equation (9),

$$\mathrm{lb}(w) = \sum_{i=1}^{n} (\max\{w_1, \ldots, w_{i-1}\} - w_i) = \sum_{i=1}^{n} (\max\{v_1, \ldots, v_{i-1}\} - v_i) = \mathrm{lb}(v).$$

The special case of $v = \mathrm{inc}(w)$ now follows from the definition of the function. $\square$

We wish to show that the generating function $\mathrm{RS}_n(1212)$ discussed in Section 5 is equal to $\mathrm{LB}_n(1221)$. The proof will be similar to that of Theorem 5.7 in that we will construct a bijection β from two-colored Motzkin paths length $n-1$ to $R_n(1221)$ which maps area to lb. The map β will not be difficult to describe. However, proving that β is a bijection will require a detailed argument. We define a map $\alpha : R_k(1221) \to R_{k+2}(1221)$ and provide the following lemma to assist us. This map will be useful when discussing two-colored Motzkin paths which are obtained from a smaller path by prepending an up step and appending a down step. Given any $v \in R_k(1221)$ we define $\bar{v} = \bar{v}_1\bar{v}_2 \ldots \bar{v}_k$ such that

$$\bar{v}_i = \begin{cases} v_i + 1 & \text{if } v_i \text{ is a first occurrence,} \\ v_i & \text{else.} \end{cases} \tag{10}$$

It is not hard to see that $u = 1\bar{v}1$ is an RGF, but it may not avoid 1221, so we define

$$\alpha(v) = \mathrm{inc}(u)$$

which is in $R_{k+2}(1221)$ by Lemma 6.2. For example, if $v = 1212344$ will have $u = 1\bar{v}1 = 123124541$ and $\alpha(v) = 123114524$.

Lemma 6.4. *For $k \geq 0$ the map $\alpha : R_k(1221) \to R_{k+2}(1221)$ is an injection. Furthermore, the image of α is precisely the $w \in R_{k+2}(1221)$ satisfying the following three properties.*

(i) *The word w has more than one 1 and ends in a repeated letter.*
(ii) *If w_i is a repeated letter then $w_i < \max\{w_1, \ldots, w_{i-1}\}$.*
(iii) *If, for $i \leq j$, we have w_{i-1} and w_{j+1} are repeated letters with $w_iw_{i+1} \ldots w_j$ all first occurrences then $w_{j+1} < w_i - 1$.*

Proof. We will start by showing that α is injective. Given a $v \in R_k(1221)$, consider $u = 1\bar{v}1$. We can easily recover $\bar{v}$ from u by removing the first and last 1, and can further recover v by decreasing all left to right maxima in $\bar{v}$ by one. We finish showing that α is injective by recovering u from $w = \mathrm{inc}(u)$. Note that since v avoids 1221, its subword r of all repeated letters is weakly increasing. The subword of all repeated letters in $u = 1\bar{v}1$

is then $r1$. Making this subword increasing results in the subword of all repeated letters in w being $1r$. We can thus recover u by replacing $1r$ in w by $r1$.

Next, we show that any w in the image of α satisfies all three properties. Since $u = 1\bar{v}1$ has more than one 1 and ends in a repeated letter, the RGF $w = \mathrm{inc}(u)$ does as well. Property (i) is thus satisfied. Next we show property (ii) by first showing that u satisfies property (ii). If v_i is a repeated letter then we always have $v_i \leq \max\{v_1, \ldots, v_{i-1}\}$. Since we increased all first occurrences to get $\bar{v}$ and left the repeated letters the same we have $\bar{v}_i < \max\{\bar{v}_1, \ldots, \bar{v}_{i-1}\}$. And clearly the two new ones in u do not change this inequality. As previously noted, the value in the place of a given repeated letter can only get weakly smaller in passing from u to $w = \mathrm{inc}(u)$. And since left to right maxima don't change, w also satisfies property (ii). Lastly, we will show property (iii). Consider the situation where $w_i w_{i+1} \ldots w_j$ are all first occurrences but w_{i-1} and w_{j+1} are repeated letters. But then w_{j+1} was in position $i-1$ in u which is also a position in $\bar{v}$. And the element in position i of u is w_i which is a left to right maximum. Since left to right maxima in v were increased by one in passing from v to $\bar{v}$ we have $w_{j+1} < w_i - 1$ as desired.

Finally, we show that if $w \in R_{k+2}(1221)$ satisfies (i)–(iii) then it must be in the image of α. By the first half of (i) and Lemma 6.2, the subword of repeated letters in w is $1r$ for some weakly increasing word r. Replace $1r$ in w with $r1$ to get u. Property (iii) assures us that all repeated letters in w remain repeated letters in u. Because w satisfies the second half of property (i), we know that u has the form $u = 1\bar{v}1$ for some word $\bar{(v)}$. Note that the repeated letters in $1\bar{v}1$ satisfy (ii) because w originally satisfied (ii) and (iii). Also, the subword of repeated letters in $1\bar{v}$ is r which is weakly increasing. Because the repeated letters in $1\bar{v}$ satisfy (ii) when we decrease all the first occurrences of $\bar{v}$, excluding ones, by 1 to get a word v it will be an RGF and r will be the subword of repeated letters in v. Because r is weakly increasing $v \in R_k(1221)$. It is not hard to see that $\alpha(v) = w$. $\square$

Our goal is to define a map $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ which maps area to lb. Before we define β we discuss a partition of the region under $R = s_1 \ldots s_{n-1} \in \mathcal{M}^2_{n-1}$ which will aid us in this task. Fig. 4 gives an example of this process where different shadings indicate parts of the partition. Recall that $l(s_i)$ is the level, or smallest y-value, of s_i. If $s_i = D$, we define $A(s_i)$ to be equal to the area in the same row between s_i and its paired up step but excluding the area under other down steps or a-steps. In Fig. 4, $A(s_5) = 1$, $A(s_8) = A(s_{12}) = 2$, and $A(s_9) = 5$. The area under R can be partitioned as follows. The rectangle under an a-step s_i will be a part with area $l(s_i)$. For example, in the figure we have the area $l(s_4) = 2$. Our other parts will be associated to down steps. Given a down step s_i, its part will consist of the region counted by $A(s_i)$ together with the rectangle of squares under the down step whose area is given by $l(s_i)$, for a total area of $A(s_i)+l(s_i)$. Returning to our example, steps s_5, s_8, s_9, and s_{12} contribute total areas $2, 3, 5$, and 2, respectively. Since these partition the full region under R we have

$$\mathrm{area}(R) = \sum_{s_i=a} l(s_i) + \sum_{s_i=D} (A(s_i) + l(s_i)). \tag{11}$$

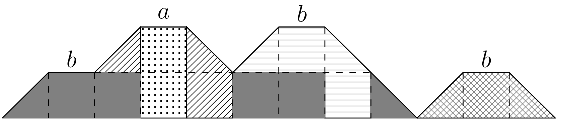

Fig. 4. The area decomposition of a two-colored Motzkin path.

Next we will define a map $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ such that $\text{area}(R) = \text{lb}(\beta(R))$. Before we define $\beta(R)$ we define an RGF, $v(R) = v_1 \dots v_n$, by letting $v_1 = 1$ and

$$v_{i+1} = \begin{cases} \max\{v_1, \dots, v_i\} + 1 & \text{if } s_i = U \text{ or } b, \\ \max\{v_1, \dots, v_i\} - l(s_i) & \text{if } s_i = a, \\ \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i) & \text{if } s_i = D, \end{cases}$$

for $i \geq 0$. For the two-colored Motzkin path R in Fig. 4 we have $v(R) = 1234225631786$.

A comparison of the first case in the definition of v with the other two shows that the left to right maxima of v are consecutive integers starting at 1. So to show that v is an RGF we only have to prove that $v_{i+1} > 0$ for all $s_i \in \{a, D\}$. Note that for all $i \geq 1$ we have that $\max\{v_1, \dots, v_i\}$ is equal to one more than the number of b-steps plus the number of up steps in the first $i-1$ steps. The level $l(s_i)$ of any horizontal step is at most the number of previous up steps, so for $s_i = a$ we have $v_{i+1} = \max\{v_1, \dots, v_i\} - l(s_i) > 0$. Note that the area counted by $A(s_i)$ between $s_i = D$ and its corresponding up step is at most the number of up steps plus b-steps between and including the paired up and down steps. Also, the level of the down step is at most the number of up steps strictly before its paired up step. All together $A(s_i) + l(s_i)$ is at most the number of up steps and b-steps in the first $i - 1$ steps. As a result, for $s_i = D$ we have $v_{i+1} = \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i) > 0$. Hence, v is an RGF. However, $v(R)$ may not avoid 1221, so we define

$$\beta(R) = \text{inc}(v(R))$$

which avoids 1221 by Lemma 6.2. For the two-colored Motzkin path R in Fig. 4 we have $\beta(R) = 1234125623786$.

Next we show that $\text{area}(R) = \text{lb}(v)$ which will imply that $\text{area}(R) = \text{lb}(\beta(R))$ by Lemma 6.3. It is easy to see that $\text{lb}(v_1) = 0$ and if s_i is b or U then $\text{lb}(v_{i+1}) = 0$. Next consider $s_i = a$ so $v_{i+1} = \max\{v_1, \dots, v_i\} - l(s_i)$. By equation (9), we have $\text{lb}(v_{i+1}) = l(s_i)$. Lastly, if $s_i = D$ then $v_{i+1} = \max\{v_1, \dots, v_i\} - A(s_i) - l(s_i)$. By equation (9) again, $\text{lb}(v_{i+1}) = A(s_i) + l(s_i)$. As a result

$$\text{lb}(v) = \sum_{s_i = a} l(s_i) + \sum_{s_i = D} (A(s_i) + l(s_i)) = \text{area}(R)$$

by equation (11).

We now show that the β map behaves nicely with respect to two of the usual decompositions of Motzkin paths.

Lemma 6.5. *Let P and Q be two-colored Motzkin paths with $\beta(P) = x$ and $\beta(Q) = 1y$. The map β has the following properties.*

(1) $\beta(PQ) = x(y + \max(x) - 1)$.
(2) $\beta(UPD) = \alpha(x)$.

Proof. To prove statement (1), we first claim that

$$v(PQ) = v(P)(q + \max(v(P)) - 1)$$

where q is $v(Q)$ with its initial 1 deleted. It is clear from the definition of v that the first $|P| + 1$ positions of $v(PQ)$ are $v(P)$. Also by definition of v, the first occurrences other than the initial 1 are in bijection with the union of the up steps and b-steps. It follows that the subword of first occurrences in the last $|Q|$ positions of $v(PQ)$ is the same as the corresponding subword in q with all elements increased by $\max(v(P)) - 1$. Thus the maximum value in any prefix of $v(PQ)$ ending in these positions is increased over the corresponding maximum in q by this amount. Furthermore, the areas and levels of down steps and a-steps in Q in that portion of PQ are the same since P ends on the x-axis. So, using the definition of v for these types of steps, the last $|Q|$ positions of $v(PQ)$ are exactly $q' = q + \max(v(P)) - 1$. To prove the equation for β, it suffices to show that the inc operator only permutes elements within $v(P)$ and within q'. But this is true because all elements of q' are greater than or equal to those of $v(P)$.

To prove the second statement, first consider $v := v(P) = v_1 \dots v_k$ and $u := v(UPD) = u_1 \dots u_{k+2}$. We claim that $u = 1\bar{v}1$. Clearly u begins with a 1. To see it must also end with 1, note that since the last step of $UPD = s_1 \dots s_{k+1}$ is a down step and this path does not touch the axis between its initial and final points, we have $l(s_{k+1}) = 0$ and $A(s_{k+1})$ is the total number of up steps and b-steps in UPD. It now follows from the definition of the map v and our interpretation of the maximum of a prefix that $u_{k+2} = 1$. Let u' be u with its initial and final 1's removed. To see that $u' = \bar{v}$, first note that every step of UPD except the first is preceded by one more up step than the corresponding step in P. It follows every first occurrence of v is increased by one in passing to u'. But the area under each a-step and under each down step also increases by one during that passage. So the differences defining the v-map in such cases will stay the same for these repeated entries. It should now be clear that $u' = \bar{v}$. It follows immediately that $\beta(UPD) = \mathrm{inc}(1\bar{v}1) = \alpha(x)$. □

Before we show that β is a bijection, we will need a method for determining from the image of a path where that path first returns to the x-axis. The following lemma will provide the key.

Lemma 6.6. *Given paths $P \in \mathcal{M}^2_{k-3}$ where $k \geq 3$ and Q, the word $\beta(UPDQ) = w$ has w_k as the right-most repeated letter such that $w_1 \dots w_k$ satisfies all three properties in Lemma 6.4.*

Proof. Given a path $R = UPDQ \in \mathcal{M}^2_{n-1}$ as stated, by Lemma 6.5 we know that if we write $\beta(Q) = 1q$ then

$$w = \beta(R) = \alpha(\beta(P))(q + m - 1) \tag{12}$$

where $m = \max(\alpha(\beta(P)))$. Lemma 6.4 implies that the prefix $w_1 \dots w_k = \alpha(\beta(P))$ satisfies all three properties. So it suffices to show that if there exists another repeated letter w_i after w_k then $w_1 \dots w_i$ fails property (ii) or property (iii). In particular, it suffices to show such a failure for the prefix where w_i is the next repeated letter after w_k since any other prefix under consideration contains $w_1 \dots w_i$.

If $i = k+1$ then, since every element of q is increased by $m-1$ and w_{k+1} is repeated, we must have $w_{k+1} = m = \max\{w_1, \dots, w_k\}$, contradicting property (ii). If instead $i > k+1$ then w_{k+1} is a first occurrence and $w_{k+1} = \max\{w_1, \dots, w_k\} + 1 = m + 1$. By definition of w_i, we have that $w_{k+1}, \dots, w_{i-1}$ are all first occurrences with w_k and w_i repeated letters. Note that all elements in q were at least 1 and then increased by $m-1$, so we must have $w_i \geq m = w_{k+1} - 1$ which contradicts property (iii). $\square$

It will be helpful for us to be able to refer to the special repeated letter mentioned in the lemma above. So, given an RGF $w = w_1 \dots w_n$, if there exists a right-most repeated letter w_k such that $w_1 \dots w_k$ satisfies all three properties in Lemma 6.4 then we will say that w_k *breaks the word* w. Note that if such a repeated letter exists, its index k is unique.

Theorem 6.7. *The map $\beta : \mathcal{M}^2_{n-1} \to R_n(1221)$ is a bijection and* $\operatorname{area}(R) = \operatorname{lb}(\beta(R))$.

Proof. We have already shown that β is a well-defined map and that $\operatorname{area}(R) = \operatorname{lb}(\beta(R))$. Since $|\mathcal{M}^2_{n-1}| = C_n = |R_n(1221)|$, to show β is a bijection it suffices to show β is injective. We prove this by induction on n. It is easy to see that β is an injection for $n \leq 2$. We now assume that $n > 2$ and $\beta : \mathcal{M}^2_{k-1} \to R_k(1221)$ is injective for all $k < n$.

We will discuss three cases for paths $R \in \mathcal{M}^2_{n-1}$ and in each case we will show that R maps to an RGF distinct from the other RGFs in that case and also from the RGFs in previous cases.

First consider all paths R which start with an a-step so that $R = aQ$ for some path Q. By Lemma 6.5, we have $\beta(R) = 11y$ where $\beta(Q) = 1y$. Injectivity of the map now follows from the fact that, by induction, it is injective on paths Q of length $n-2$.

Our second case consists of paths R of the form $R = bQ$. Now $\beta(R) = 12(y+1)$ with y as above. Clearly these are distinct from the words in the previous paragraph and injectivity within this case follows by induction as before.

For the last case, consider all paths R which start with an up step so we can write $R = UPDQ$ for paths $P \in \mathcal{M}^2_{k-3}$ and Q where $k \geq 3$. By Lemma 6.5 we have equation (12), and by Lemma 6.6 the repeated letter w_k breaks the word w. Note that because $\alpha(\beta(P)) = w_1 \dots w_k$ satisfies property (i) in Lemma 6.4, w has more than one 1 and so can not agree with a word from the second case above. But since R starts with an up step, w starts with the prefix 12 and so can not be a word from the first case. Finally, by uniqueness of the index of w_k, the injectivity of the map α, and induction the word w is uniquely determined among all words in this case. This finishes the proof that β is injective. $\square$

Combining the previous result with Corollary 5.6 and the definition of $M_n(q)$ in equation (8) we have the following corollary.

Corollary 6.8. *We have, for all $n \geq 1$,*

$$\mathrm{LB}_n(1221) = \mathrm{RS}_n(1212) = M_{n-1}(q). \quad \square$$

Since it may be of interest, we will now give an explicit way to calculate β^{-1}, leaving the proof that it is indeed a well-defined inverse for β to the reader. Given a word $w \in R_n(1221)$ we have an associated sequence $\mathrm{lb}(w_1)\,\mathrm{lb}(w_2)\dots\mathrm{lb}(w_n)$ recording lb for each letter. We use this to inductively describe the sequence of heights h at the end points of the steps in the associated two-colored Motzkin path. Let $h^{(1)} = \mathrm{lb}(w_1)$,

$$h^{(i)} = h^{(i-1)}_1 \dots h^{(i-1)}_{i-\mathrm{lb}(w_i)-1}(h^{(i-1)}_{i-\mathrm{lb}(w_i)} + 1)\dots(h^{(i-1)}_{i-1} + 1)0$$

and we let $h = h^{(n)}$. From h we can determine the up-steps and the down-steps of the path by letting the left endpoint of s_i have height h_i for $i < n$. And to determine the colors, if we have a horizontal step because $h_{i-1} = h_i$ and w_i is a repeated letter then s_{i-1} is an a-step and otherwise it is a b-step.

6.3. Combinations with other patterns

Next we consider the RGFs which avoid 1221 and another length three pattern. Since 121 and 122 are subwords of 1221 these cases are not interesting, so we will focus on 111, 112, and 123.

Theorem 6.9. *We have for $L_n := \mathrm{LB}_n(111, 1221)$ that $L_0 = L_1 = 1$ and, for $n \geq 2$, and*

$$L_n = L_{n-1} + L_{n-2} + \sum_{k=1}^{n-2} q^k L_{k-1}L_{n-k-1}$$

Proof. Let $\mathcal{N}_n$ be the collection of two-colored Motzkin paths $R \in \mathcal{M}^2_n$ such that $\beta(R)$ avoids 111. Define $N_{-1}(q) = 1$ and, for $n \geq 0$,

$$N_n(q) = \sum_{R \in \mathcal{N}_n^2} q^{\text{area}(R)}.$$

By Theorem 6.7 we only need to show that $N_n(q) = \text{LB}_{n+1}(111, 1221)$ satisfies an equivalent recurrence and initial conditions. By definition $N_{-1}(q) = 1$, and $N_0(q) = 1$ because of the empty path. So we wish to show that for $n \geq 1$

$$N_n(q) = N_{n-1}(q) + N_{n-2}(q) + \sum_{k=0}^{n-2} q^{k+1} N_{k-1}(q) N_{n-k-2}(q). \tag{13}$$

We partition $\mathcal{M}_n^2$ as in the proof of Theorem 6.7:

$$\begin{aligned} X &= \{R = aQ : \ Q \in \mathcal{M}_{n-1}^2\}, \\ Y &= \{R = bQ : \ Q \in \mathcal{M}_{n-1}^2\}, \\ Z &= \{R = UPDQ : \ P \in \mathcal{M}_k^2, \ Q \in \mathcal{M}_{n-k-2}^2 \text{ and } k \in [0, n-2]\}. \end{aligned}$$

We claim that when we restrict this partition to paths in $\mathcal{N}_n$ we have

$$\begin{aligned} X_{\mathcal{N}} &= \{R = abQ : \ Q \in \mathcal{N}_{n-2}\}, \\ Y_{\mathcal{N}} &= \{R = bQ : \ Q \in \mathcal{N}_{n-1}\}, \\ Z_1 &= \{R = UDQ : \ Q \in \mathcal{N}_{n-2}\}, \\ Z_2 &= \{R = UbPDQ : \ P \in \mathcal{N}_{k-1}, \ Q \in \mathcal{N}_{n-k-2}, \text{ and } k \in [n-2]\}, \end{aligned}$$

where the set Z breaks into two subsets. From the second partition we will be able to deduce the desired recursion.

Consider a path $R = aQ \in X$. We claim that $\beta(R)$ avoids 111 if and only if $Q = bQ'$ for $Q' \in \mathcal{N}_{n-2}$ which will show that X restricts to $X_{\mathcal{N}}$. If we write $\beta(Q) = 1y$ we have $\beta(R) = 11y$. The word $\beta(R)$ avoids 111 if and only if the word y has no 1's and at most two copies of every other number. Note that the second case considered in Theorem 6.7 contained all paths which started with a b-step and that these paths were mapped bijectively to words with exactly one 1. It is also clear that $y = \beta(Q') + 1$ has at most two copies of every number greater than one if and only if the same is true of $\beta(R)$. The claim now follows. Because $\text{area}(R) = \text{area}(Q')$ summing over all paths in this case gives us the term $N_{n-2}(q)$.

If instead $R = bQ \in Y$ then, using that notation of Lemma 6.5, $\beta(R) = 12(y + 1) = 1(\beta(Q) + 1)$. So $\beta(R)$ avoids 111 if and only if $\beta(Q)$ does. It follows that Y restricts to $Y_{\mathcal{N}}$. Because $\text{area}(R) = \text{area}(Q)$ summing over all paths in this case gives us the term $N_{n-1}(q)$.

Next, we consider paths $R = UPDQ$ from the third set, Z. First consider the case where P has length 0 so $R = UDQ$. We want to prove that $\beta(R)$ avoids 111 if and only if $\beta(Q)$ avoids 111 since this will show that the collection of paths in Z with $|P| = 0$

restricts to Z_1. If we write $\beta(Q) = 1y$ we have $\beta(R) = 121(y+1)$. Thus $\beta(Q)$ avoids 111 if and only if $\beta(R)$ does and the restriction is as claimed. Because $\text{area}(R) = 1 + \text{area}(Q)$ summing over all paths in this case gives us the term $qN_{n-2}(q)$ which is the $k = 0$ term in equation (13).

Lastly, consider a path $R = UPDQ$ with $|P| = k \in [n-2]$ which are the remaining paths in Z. We will show that $\beta(R)$ avoids 111 if and only if $P = bP'$ and both the words $\beta(P')$ and $\beta(Q)$ avoid 111. This will show that the remaining paths in Z restrict to Z_2 in the second partition. First we make an observation about $\alpha(\beta(P))$. Let $m = \max(\beta(P))$ and $\{1^{s_1}, \ldots, m^{s_m}\}$ be the multiset of all letters in $\beta(P)$. The map α increases all first occurrences by one and adds two 1's but otherwise doesn't affect the collection of letters. So the multiset of letters in $\alpha(\beta(P))$ is $\{1^{s_1+1}, \ldots, m^{s_m}, m+1\}$. If we write $\beta(Q) = 1y$ then we have $\beta(R) = \alpha(\beta(P))(y+m)$ since $m = \max(\alpha(\beta(P))) - 1$. If $\{1^{t_1}, \ldots, \bar{m}^{t_{\bar{m}}}\}$ is the multiset of letters in $\beta(Q)$ then the multiset of letters in $\beta(R)$ is $\{1^{s_1+1}, \ldots, m^{s_m}, (m+1)^{t_1}, \ldots, (m+\bar{m})^{t_{\bar{m}}}\}$. So $\beta(R)$ avoids 111 if and only if there are at most two of any element in this set which is equivalent to $s_1 = 1$, $s_i \leq 2$ for $i > 1$, and $t_i \leq 2$ for all $i \geq 1$. Further this implies that $\beta(R)$ avoids 111 if and only if $Q \in \mathcal{N}_{n-k-2}$ and $\beta(P)$ has exactly one 1 and avoids 111. Just as in our first case, $\beta(P)$ has exactly one 1 and avoids 111 if and only if $P = bP'$ for some $P' \in \mathcal{N}_{k-1}$. Because $\text{area}(R) = \text{area}(P') + \text{area}(Q) + k + 1$ summing over all paths in this case gives us the term $q^{k+1}N_{k-1}(q)N_{n-k-2}(q)$ for $k > 0$. This completes the proof of the theorem. $\square$

The next two avoidance classes can be characterized by a combination of Theorem 1.2 and Lemma 6.2. The proofs are straightforward and so not included.

Proposition 6.10. *We have*

$$R_n(112, 1221) = \{12\ldots mk^{n-m} : \ k \in [m]\}.$$

So for $n \geq 0$ we have

$$F_n(112, 1221) = (rs)^{\binom{n}{2}} + \sum_{m=1}^{n-1}\sum_{k=1}^{m} q^{(n-m)(m-k)} r^{\binom{m}{2}+(n-m)(k-1)} s^{\binom{m}{2}} t^{m-k}. \quad \square$$

Corollary 6.11. *We have, for $n \geq 0$,*

1. $\text{LB}_n(112, 1221) = 1 + \displaystyle\sum_{m=1}^{n-1}\sum_{k=1}^{m} q^{(n-m)(m-k)}$,
2. $\text{LS}_n(112, 1221) = q^{\binom{n}{2}} + \displaystyle\sum_{m=1}^{n-1}\sum_{k=1}^{m} q^{\binom{m}{2}+(n-m)(k-1)}$,
3. $\text{RB}_n(112, 1221) = q^{\binom{n}{2}} + \displaystyle\sum_{m=1}^{n-1} mq^{\binom{m}{2}}$, *and*

4. $\mathrm{RS}_n(112, 1221) = 1 + \sum_{i=1}^{n-1} iq^{n-i-1}$. □

Proposition 6.12. *We have*

$$R_n(123, 1221) = \{1^n, 11^i21^j2^k : \ i + j + k = n - 2, \ \text{and } i, j, k \geq 0\}.$$

So for $n \geq 0$ we have, using the truth function $\chi(S) = 1$ if S is true or 0 if S is false,

$$F_n(123, 1221) = 1 + \sum_{\substack{i+j+k=n-2 \\ i,j,k \geq 0}} q^j r^{k+1} s^{i+1+j\cdot\chi(k>0)} t^{\chi(j>0)}. \quad \square$$

Corollary 6.13. *We have, for $n \geq 0$,*

1. $\mathrm{LB}_n(123, 1221) = 1 + \sum_{j=0}^{n-2} (n - j - 1)q^j$,
2. $\mathrm{LS}_n(123, 1221) = 1 + \sum_{k=0}^{n-2} (n - k - 1)q^{k+1}$,
3. $\mathrm{RB}_n(123, 1221) = 1 + q^{n-1} + \sum_{k=1}^{n-2} (k + 1)q^k$, *and*
4. $\mathrm{RS}_n(123, 1221) = n + \binom{n-1}{2} q$. □

6.4. More about the pattern 1212

It turns out that the generating function $\mathrm{LB}_n(1212)$ is also equal to $M_{n-1}(q)$. Instead of showing this directly, we prove that $\mathrm{LB}_n(1212) = \mathrm{LB}_n(1221)$ and then Corollary 6.8 completes the proof. In the process we also show $\mathrm{LS}_n(1212) = \mathrm{LS}_n(1221)$.

Proposition 6.14. *The restriction* inc : $R_n(1212) \to R_n(1221)$ *is a bijection which preserves* lb *and* ls.

Proof. By Lemma 6.2 we have $\mathrm{lb}(w) = \mathrm{lb}(\mathrm{inc}(w))$. This map also preserves ls because w and $\mathrm{inc}(w)$ are rearrangements of each other and $\mathrm{ls}(w_i) = w_i - 1$ for any RGF w.

Now we only need to show that inc : $R_n(1212) \to R_n(1221)$ is bijective. Since $|R_n(1212)| = C_n = |R_n(1221)|$ it suffices to show the map is injective. Assume that $v = v_1v_2 \ldots v_n$ and $w = w_1 \ldots w_n$ are two distinct words which avoid 1212, but $\mathrm{inc}(v) = \mathrm{inc}(w)$. This means that v and w share the same positions of first occurrences, and the same multiset of repeated letters. But since $v \neq w$ there is then a smallest index $i \geq 1$ such that $v_1 \ldots v_{i-1} = w_1 \ldots w_{i-1}$ but $v_i \neq w_i$. Without loss of generality let $v_i = x$, $w_i = y$, and $x < y$. We have noted that v and w have their first occurrences at the same indices, so v_i and w_i must be repeated letters. Since w is an RGF, the first occurrence of x and y must occur before w_i, so v also has the subword xy before v_i. However, because v and w have the same collection of repeated letters and agree up to position $i-1$, the y which is w_i in w must occur some time after v_i in v. This means that v has the subword $xyxy$ contradicting Lemma 5.2. $\square$

Corollary 6.15. *For $k \geq 0$ we have*

$$F_n(1212; q, r, 1, 1) = F_n(1221; q, r, 1, 1),$$
$$F_n(1^k, 1212; q, r, 1, 1) = F_n(1^k, 1221; q, r, 1, 1),$$

and

$$F_n(12\ldots k, 1212; q, r, 1, 1) = F_n(12\ldots k, 1221; q, r, 1, 1).$$

Proof. The bijection f in Proposition 6.14 preserves the number of times any integer appears and preserves the maximum integer which appears. The equalities follow from this fact. $\square$

Using Proposition 5.11, and Corollaries 5.6, 6.8, 6.13, and 6.15 we have the following equalities which summarize the results in this section.

Corollary 6.16. *We have, for $n \geq 0$,*

$$\mathrm{LB}_n(1212) = \mathrm{RS}_n(1212) = \mathrm{LB}_n(1221) = M_{n-1}(q),$$
$$\mathrm{LS}_n(1212) = \mathrm{LS}_n(1221),$$
$$\mathrm{LB}_n(111, 1212) = \mathrm{LB}_n(111, 1221),$$
$$\mathrm{LS}_n(111, 1212) = \mathrm{LS}_n(111, 1221),$$
$$\mathrm{LB}_n(123, 1212) = \mathrm{RS}_n(123, 1212) = \mathrm{LB}_n(123, 1221),$$

and

$$\mathrm{LS}_n(123, 1212) = \mathrm{RB}_n(123, 1212) = \mathrm{LS}_n(123, 1221). \quad \square$$

We note that Simion [22] also proved $\mathrm{LB}_n(1212) = \mathrm{RS}_n(1212)$ by different means. In addition, she showed the following.

Theorem 6.17 *([22]). We have, for $n \geq 0$,*

$$LS_n(1212) = RB_n(1212). \quad \square$$

7. Comments and open problems

We list some further possible lines of research in the hopes that the reader may be interested in pursuing them.

(1) **Longer patterns.** In Sections 4, 5, and 6 we have begun the study of patterns of length four or more, but there are almost certainly more interesting results for such patterns. For example, for noncrossing partitions it would be interesting to see if the polynomial $\mathrm{LS}_n(1212) = \mathrm{RB}_n(1212)$ can be viewed as the generating function for a statistic over two-colored Motzkin paths. And here is a specific conjecture for nonnesting patterns.

Conjecture 7.1. *The coefficients of* $\mathrm{RB}_n(1221)$ *stabilize in the following sense. Given* k *there is a bound* N_k *such that for* $n \geq N_k$ *the coefficient of* q^k *in* $\mathrm{RB}_n(1221)$ *is constant.*

(2) **Vincular patterns.** In the theory of permutation patterns a *vincular* or *generalized* pattern is one where copies of the pattern in a larger permutation are required to have certain elements adjacent. One can indicate such elements by underlining them. For example, a copy of the pattern 21 is an inversion while a copy of the pattern $\underline{21}$ is a descent. In [2], Babson and Steingrímsson initiated the study of such patterns and showed that a wide array of well-known permutation statistics could be realized as linear combinatorics of functions counting vincular patterns. One can also consider patterns where certain integers which are numerically adjacent in the pattern must be numerically adjacent in the copy and indicate these by an overline. So $\overline{21}$ would count inversions consisting of an element k followed by $k-1$. And, of course, one could combine positional and numerical adjacency. It seems probable that studying vincular RGF patterns would yield interesting enumerative results.

(3) **Equidistribution.** In their original paper, Wachs and White [25] proved that lb and rs are equidistributed (have the same generating function) over the set of all RGFs of length n with maximum m. They also showed that ls and rb are equidistributed over the same set of RGFs. We have seen similar behavior in Theorems 2.1, 2.5, 2.8, and 6.17 as well as Corollaries 3.4 and 6.16. It would be very interesting to derive some of these results from more general theorems which would guarantee equidistribution for a large number of avoidance classes.

(4) **Mahonian pairs.** When considering st-Wilf equivalence, one has a single statistic which has the same generating function over two different avoidance classes. When considering equidistribution, one has two different statistics which have the same generating function over a given avoidance class. Obviously, one could generalize both notions by considering one statistic on an avoidance class and a second statistic on another class. For the permutation statistics given by the major index, maj, and inversion number, inv, this concept was first studied by Sagan and Savage [21]. Such pairs of statistics and

classes were called *Mahonian pairs* since maj and inv both have the Mahonian distribution over the full symmetric group. In the present work, we have found such equalities in the results cited in (3) as well as in Theorem 2.9 and Corollary 6.16. Again, a more general explanation of when such identities occur would be desirable.

(5) **Other statistics.** There are other statistics related to the four we have been studying. Given an integer sequence $w = w_1 \dots w_n$, Simion and Stanton [23] considered a statistic counting smaller elements both to the left and the right of each w_j by letting

$$\operatorname{lrs}(w_j) = \#\{x < w_j \ : \ \text{there are } i < j < k \text{ with } w_i = w_k = x\}$$

and $\operatorname{lrs}(w) = \sum_j \operatorname{lrs}(w_j)$. Note that if w is an RGF then $\operatorname{lrs}(w) = \operatorname{rs}(w)$. They also looked at an analogous statistic for counting bigger elements, as well as refinements of both statistics obtained by restricting them to certain elements of w related to first occurrences and repeated elements. Their motivation came from studying a generalization of the Laguerre polynomials. In the process, they obtained results about these statistics on noncrossing and nonnesting RGFs. It would be interesting to investigate these statistics in relation to other patterns.

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References

[1] Drew Armstrong, Generalized noncrossing partitions and combinatorics of Coxeter groups, Mem. Amer. Math. Soc. 202 (949) (2009), x+159.
[2] Eric Babson, Einar Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, Sém. Lothar. Combin. 44 (2000), Art. B44b, 18 pp.
[3] Jonathan Bloom, Dan Saracino, Pattern avoidance for set partitions à la klazar, Preprint, arXiv:1511.00192v2.
[4] William Y.C. Chen, Alvin Y.L. Dai, Theodore Dokos, Tim Dwyer, Bruce E. Sagan, On 021-avoiding ascent sequences, Electron. J. Combin. 20 (1) (2013), Paper 76, 6 pp.
[5] Szu-En Cheng, Sergi Elizalde, Anisse Kasraoui, Bruce E. Sagan, Inversion polynomials for 321-avoiding permutations, Discrete Math. 313 (22) (2013) 2552–2565.
[6] Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Lindsey Reppuhn, Bruce E. Sagan, Set partition patterns and statistics, Discrete Math. 339 (1) (2016) 1–16.
[7] Theodore Dokos, Tim Dwyer, Bryan P. Johnson, Bruce E. Sagan, Kimberly Selsor, Permutation patterns and statistics, Discrete Math. 312 (18) (2012) 2760–2775.
[8] Brian Drake, Limits of areas under lattice paths, Discrete Math. 309 (12) (2009) 3936–3953.
[9] Paul Duncan, Einar Steingrímsson, Pattern avoidance in ascent sequences, Electron. J. Combin. 18 (1) (2011), Paper 226, 17 pp.
[10] Adam M. Goyt, Avoidance of partitions of a three-element set, Adv. in Appl. Math. 41 (1) (2008) 95–114.
[11] Adam M. Goyt, David Mathisen, Permutation statistics and q-Fibonacci numbers, Electron. J. Combin. 16 (1) (2009), Research Paper 101, 15 pp.
[12] Adam M. Goyt, Bruce E. Sagan, Set partition statistics and q-Fibonacci numbers, European J. Combin. 30 (1) (2009) 230–245.

[13] Vít Jelínek, Toufik Mansour, On pattern-avoiding partitions, Electron. J. Combin. 15 (R39) (2008) 1–52.
[14] Kendra Killpatrick, On the parity of certain coefficients for a q-analogue of the Catalan numbers, Electron. J. Combin. 19 (4) (2012), Paper 27, 7 pp.
[15] Jang Soo Kim, Front representation of set partitions, SIAM J. Discrete Math. 25 (1) (2011) 447–461.
[16] Martin Klazar, On *abab*-free and *abba*-free set partitions, European J. Combin. 17 (1) (1996) 53–68.
[17] Martin Klazar, Counting pattern-free set partitions. I. A generalization of Stirling numbers of the second kind, European J. Combin. 21 (3) (2000) 367–378.
[18] Martin Klazar, Counting pattern-free set partitions. II. Noncrossing and other hypergraphs, Electron. J. Combin. 7 (2000), Research Paper 34, 25 pp.
[19] Toufik Mansour, Mark Shattuck, Pattern-avoiding set partitions and Catalan numbers, Electron. J. Combin. 18 (2) (2011), Paper 34, 18 pp.
[20] Bruce E. Sagan, Pattern avoidance in set partitions, Ars Combin. 94 (2010) 79–96.
[21] Bruce E. Sagan, Carla D. Savage, Mahonian pairs, J. Combin. Theory Ser. A 119 (3) (2012) 526–545.
[22] Rodica Simion, Combinatorial statistics on noncrossing partitions, J. Combin. Theory Ser. A 66 (2) (1994) 270–301.
[23] Rodica Simion, Dennis Stanton, Octabasic Laguerre polynomials and permutation statistics, J. Comput. Appl. Math. 68 (1–2) (1996) 297–329.
[24] Gérard Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in: Orthogonal Polynomials and Applications, Bar-le-Duc, 1984, in: Lecture Notes in Math., vol. 1171, Springer, Berlin, 1985, pp. 139–157.
[25] Michelle Wachs, Dennis White, p, q-Stirling numbers and set partition statistics, J. Combin. Theory Ser. A 56 (1) (1991) 27–46.