Pattern avoidance and quasisymmetric functions

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December 15, 2018

Abstract

Given a set of permutations Π, let $\mathfrak{S}_n(\Pi)$ denote the set of permutations in the symmetric group $\mathfrak{S}_n$ that avoid every element of Π in the sense of pattern avoidance. Given a subset S of $\{1, \ldots, n-1\}$, let F_S be the fundamental quasisymmetric function indexed by S. Our object of study is the generating function $Q_n(\Pi) = \sum F_{\text{Des}\,\sigma}$ where the sum is over all $\sigma \in \mathfrak{S}_n(\Pi)$ and $\text{Des}\,\sigma$ is the descent set of σ. We characterize those $\Pi \subseteq \mathfrak{S}_3$ such that $Q_n(\Pi)$ is symmetric or Schur nonnegative for all n. In the process, we show how each of the resulting Π can be obtained from a theorem or conjecture involving more general sets of patterns. In particular, we prove results concerning symmetries, shuffles, and Knuth classes, as well as pointing out a relationship with the arc permutations of Elizalde and Roichman. Various conjectures and questions are mentioned throughout.

1 Introduction

Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n] = \{1, \ldots, n\}$. We view the permutations in $\mathfrak{S}_n$ as sequences $\pi = \pi_1 \ldots \pi_n$. Given any sequence σ of k distinct integers, its *standardization* is the permutation $\text{std}\,\sigma \in \mathfrak{S}_k$ obtained by replacing the smallest element of σ by 1, the next smallest by 2, and so forth. We say that permutation σ *contains permutation π as a pattern* if there is a subsequence σ' of (not necessarily consecutive) elements of σ such that $\text{std}\,\sigma' = \pi$. For example, $\sigma = 5132746$ contains $\pi = 231$ because

of the subsequence $\sigma' = 574$. Permutation σ *avoids* π if it does not contain π as a pattern. Given a set of permutations Π we let

$$\mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids every } \pi \in \Pi\}.$$

For more information about permutation patterns, see the book of Bóna [Bón04]. Our object is to make a connection between the theory of patterns and the theory of quasisymmetric functions. First we will review some material concerning symmetric functions. Details about symmetric functions and related combinatorics such as the Robinson-Schensted map can be found in the texts of Macdonald [Mac95], Sagan [Sag01], or Stanley [Sta99].

Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a countably infinite set of commuting variables and consider the algebra of formal power series over the rationals $\mathbb{Q}[[\mathbf{x}]]$. Consider a monomial $m = x_{i_1}^{n_1} \dots x_{i_k}^{n_k}$. The *degree* of m is $\sum_i n_i$, and the *degree* of any $f \in \mathbb{Q}[[\mathbf{x}]]$ is the maximum degree of a monomial in f if the maximum exists or infinity otherwise. We say that f is *symmetric* if it is of bounded degree and invariant under permutation of variables. For example

$$f = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \dots$$

is symmetric. Let Sym_n denote the vector space of symmetric functions homogeneous of degree n. Bases for Sym_n are indexed by *partitions of n*, which are weakly decreasing sequences of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\sum_i \lambda_i = n$. If λ is a partition of n then we write $\lambda \vdash n$ or $|\lambda| = n$. The λ_i are called the *parts* of λ.

We will be particularly interested in the important Schur basis for Sym_n. Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ has an associated *Young diagram* consisting of k left-justified rows of boxes with λ_i boxes in row i. We will write our diagrams in English notation with the first row on top and often make no distinction between a partition and its diagram. Given $\lambda \vdash n$ then a *standard Young tableau (SYT)*, P, of shape λ is obtained by filling the boxes bijectively with the elements of $[n]$ so that rows and columns increase. In a *semistandard Young tableau (SSYT)*, T, of shape λ the entries are positive integers distributed so that rows weakly increase and columns strictly increase. We write $\mathrm{SYT}(\lambda)$ or $\mathrm{SSYT}(\lambda)$ for the set of standard or semistandard Young tableaux of shape λ, respectively. We also write $\mathrm{sh}\, P = \lambda$ or $\mathrm{sh}\, T = \lambda$ to indicate that P or T have shape λ. The *Schur function* corresponding to λ is

$$s_\lambda = \sum_T \prod_{i \in T} x_i.$$

For example, the set $\mathrm{SSYT}(2,1)$ consists of

$$\begin{array}{ccccccccc}
\young(11,2) & , & \young(12,2) & , & \young(11,3) & , & \young(13,3) & , & \dots, & \young(12,3) & , & \young(13,2) & , & \young(12,4) & , & \young(14,2) & , & \dots \quad (1)
\end{array}$$

resulting in

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \dots + 2x_1 x_2 x_3 + 2x_1 x_2 x_4 + \dots$$

as the corresponding Schur function. Call $f \in \mathrm{Sym}_n$ *Schur nonnegative* if its expansion in the s_λ basis has nonnegative coefficients.

Quasisymmetric functions refine symmetric functions. Call a formal power series g of bounded degree *quasisymmetric* if any two monomials $x_{i_1}^{n_1} \ldots x_{i_k}^{n_k}$ where $i_1 < \cdots < i_k$ and $x_{i_1}^{n_1} \ldots x_{i_k}^{n_k}$ where $j_1 < \cdots < j_k$ have the same coefficient in g. By way of illustration,

$$g = x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + \cdots + x_2^2 x_3 + x_2^2 x_4 + \cdots + x_3^2 x_4 + \ldots$$

is quasisymmetric, but not symmetric. Clearly every symmetric function is quasisymmetric. Quasisymmetric functions were introduced by Gessel [Ges84] in his work on Stanley's theory of P-partitions. They have since found many applications in both enumerative and algebraic combinatorics. We denote by QSym_n the vector space of quasisymmetric functions homogeneous of degree n. Bases for this vector space are indexed by *compositions of n*, which are sequences of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ summing to n. We will use the notations $\alpha \models n$ and $|\alpha| = n$ for compositions of n. Greek letters near the beginning of the alphabet will be used for compositions while those near the middle will represent partitions. There is also an important bijection between compositions of n and subsets $S \subseteq [n-1]$ given by

$$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) \mapsto \{\alpha_1, \ \alpha_1 + \alpha_2, \ \alpha_1 + \alpha_2 + \alpha_3, \ \ldots, \ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}.$$

We will sometimes go back and forth between a composition and its associated set without mention.

The basis we will be using for QSym_n was considered in Gessel's original paper. Given $S \subseteq [n-1]$, the associated *fundamental quasisymmetric function* is

$$F_S = \sum x_{i_1} x_{i_2} \ldots x_{i_n}$$

where the sum is over all indices such that $i_1 \leq i_2 \leq \cdots \leq i_n$ and $i_s < i_{s+1}$ if $s \in S$. For example, if $S = \{1\} \subseteq [2]$ then the sum would be over all $x_i x_j x_k$ with $i < j \leq k$, which gives

$$F_{\{1\}} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \ldots$$

in QSym_3. Since s_λ is symmetric, and hence quasisymmetric, it can be expanded in the F_S basis. To do so, we need the *descent set of an SYT*, P, which is

$$\mathrm{Des}\, P = \{i \mid i+1 \text{ is in a lower row than } i \text{ in } P\}.$$

For example, if

$$P = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 5 & 9 \\
\hline
3 & 4 & 7 \\
\cline{1-3}
6 & 8 \\
\cline{1-2}
\end{array}$$

then $\mathrm{Des}\, P = \{2, 5, 7\}$. Note that if P has shape $\lambda \vdash n$ then $\mathrm{Des}\, P \subseteq [n-1]$.

Theorem 1.1 ([Ges84]). *We have*

$$s_\lambda = \sum_{P \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}\, P}. \qquad \qquad \square$$

Π	$Q_n(\Pi)$ for $n \geq 3$
$\emptyset$	$\displaystyle\sum_{\lambda} f^{\lambda} s_{\lambda}$
$\{123\}$	$\displaystyle\sum_{\lambda_1 < 3} f^{\lambda} s_{\lambda}$
$\{321\}$	$\displaystyle\sum_{\lambda_1^t < 3} f^{\lambda} s_{\lambda}$
$\{132, 213\}; \{132, 312\}; \{213, 231\}; \{231, 312\}$	$\displaystyle\sum_{\lambda \text{ a hook}} s_{\lambda}$
$\{123, 132, 312\}; \{123, 213, 231\}; \{123, 231, 312\}$	$s_{1^n} + s_{2,1^{n-2}}$
$\{132, 213, 321\}; \{132, 312, 321\}; \{213, 231, 321\}$	$s_n + s_{n-1,1}$
$\{132, 213, 231, 312\}$	$s_n + s_{1^n}$
$\{123, 132, 213, 231, 312\}$	s_{1^n}
$\{132, 213, 231, 312, 321\}$	s_n

Figure 1: The Π for Theorem 1.2 along with the Schur expansions of $Q_n(\Pi)$ where $\lambda \vdash n$

Returning to $\lambda = (2,1)$, the two elements of $\mathrm{SYT}(2,1)$ are the fifth and sixth displayed tableaux in (1) with descent sets $\{2\}$ and $\{1\}$ respectively. Therefore $s_{(2.1)} = F_{\{1\}} + F_{\{2\}}$.

To combine permutation patterns and quasisymmetric functions, recall that the *descent set of a permutation* $\pi = \pi_1 \ldots \pi_n$ is

$$\mathrm{Des}\, \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1].$$

For example, $\mathrm{Des}(35716824) = \{3, 6\}$. Now given a set of pattern permutations Π we define the *pattern quasisymmetric function*

$$Q_n(\Pi) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\mathrm{Des}\, \sigma}.$$

The basic questions we wish to ask about these functions are

1. When is $Q_n(\Pi)$ symmetric for all n?

2. In that case, when is $Q_n(\Pi)$ Schur nonnegative for all n?

Note that we are asking about symmetry or Schur nonnegativity not for a single function, but rather for an infinite family of functions.

We start with $\Pi \subseteq \mathfrak{S}_3$ and will prove the following theorem for which we will need some preliminaries. If $\{123, 321\} \subseteq \Pi$ then, by the Erdős-Szekeres Theorem [ES35], $\mathfrak{S}_n(\Pi) = \emptyset$

for $n \geq 5$, which explains the hypothesis on Π. We use the notation f^λ for the number of SYT of shape λ. The *transpose* of λ is the diagram λ^t obtained by reflecting λ in the main diagonal. Then λ_1^t is the number of boxes in the first column of λ, which is also written as $\ell(\lambda)$ and called the *length* of λ. Also, a *hook* is a partition of the form $(a, 1^b)$ for nonnegative integers a, b where 1^b denotes the part 1 repeated b times. When using a partition as a subscript we often omit the parentheses.

Theorem 1.2. *Suppose* $\{123, 321\} \not\subseteq \Pi \subseteq \mathfrak{S}_3$. *The following are equivalent*

1. $Q_n(\Pi)$ *is symmetric for all* n.

2. $Q_n(\Pi)$ *is Schur nonnegative for all* n.

3. Π *is an entry in Table 1.*

We note that it is easy to compute examples to show that for any Π not listed in Table 1 we have $Q_n(\Pi)$ not being symmetric for some small value of n depending on Π. Therefore it suffices to show that the Π in the table have the claimed Schur expansions. In fact, we will show that these expansions are special cases of more general results or conjectures where Π is not restricted to $\mathfrak{S}_3$.

The rest of this paper is structured as follows. In the next section we discuss what effect reversal and complementation have on $Q_n(\Pi)$ as well as using properties of the Robinson-Schensted correspondence to derive some of the results in Table 1. In Section 3 we show how the quasisymmetric function for a shuffle of two pattern sets can be computed in terms of the ones for each individual component. Shuffles with increasing and decreasing permutations as well as full symmetric groups are used as examples. We define partial shuffles in Section 4 as certain shuffles where the increasing permutation has been removed. We conjecture that in this case $Q_n(\Pi)$ has a nice Schur expansion and prove this in a special case. Clearly if $\mathfrak{S}_n(\Pi)$ is a union of Knuth classes then $Q_n(\Pi)$ is symmetric and Schur nonnegative. In Section 5 we study when this can happen and, in particular, characterize the SYT such that the permutations avoiding its Knuth class have this property. We show in Section 6 that avoiders of the arc permutations of Elizalde and Roichman [ER14, ER15] are in bijection with the permutations avoiding a certain set of shuffles. We end with a section of comments and open questions.

2 Symmetries and the Robinson-Schensted bijection

If one views a permutation as a permutation matrix, then the dihedral group D_4 of the square acts on $\mathfrak{S}_n$. We wish to investigate whether this action tells us anything about $Q_n(\Pi)$. In particular, if this function is symmetric and one acts on Π with a dihedral symmetry then is the new quasisymmetric function symmetric? If so, what is its Schur expansion? As usual, we apply a symmetry to a set by applying it to each member of the set.

First we note that if $Q_n(\Pi)$ is a symmetric function then $Q_n(\Pi^{-1})$ need not be. For example, one can easily check that this is true if $n = 3$ and $\Pi = \{132, 312\}$. Therefore we can expect the dihedral symmetries preserving symmetry of Q_n to form a subgroup of order at most 4 in D_4. We will now show that such a subgroup exists.

It is easy to describe two of the symmetries in D_4 directly in terms of the permutations in $\mathfrak{S}_n$. The permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ has *complement* $\pi^c = n{+}1{-}\pi_1, n{+}1{-}\pi_2, \ldots, n{+}1{-}\pi_n$ where we have inserted commas between the elements for readability. Its *reversal* is $\pi^r = \pi_n \pi_{n-1} \ldots \pi_1$. For example, if $\pi = 35716824$ then $\pi^c = 64283175$ and $\pi^r = 42861753$. Complementation and reversal are two of the reflections in D_4 and so $\pi^{rc} = \pi^{cr}$ is the permutation whose matrix is obtained by rotating the matrix of π by $180°$. To state our first dihedral symmetry result, recall that the transpose λ^t of a Young diagram λ is gotten by reflecting λ in its main diagonal. We use the same notation for tableaux.

Proposition 2.1. *If $Q_n(\Pi)$ is symmetric then so is $Q_n(\Pi^c)$. In particular, if $Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda$ for certain coefficients c_λ then*

$$Q_n(\Pi^c) = \sum_\lambda c_\lambda s_{\lambda^t}.$$

Proof. Using Theorem 1.1 we can write

$$\sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\mathrm{Des}\,\sigma} = Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda = \sum_\lambda c_\lambda \sum_{P \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}\,P}.$$

Note that for any permutation $\mathrm{Des}\,\pi^c = [n-1] - \mathrm{Des}\,\pi$ and $\mathfrak{S}_n(\Pi^c) = (\mathfrak{S}_n(\Pi))^c$. Also, for any standard Young tableau $\mathrm{Des}\,P^t = [n-1] - \mathrm{Des}\,P$. Using these facts and the previous displayed equation gives

$$Q_n(\Pi^c) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{[n-1]-\mathrm{Des}\,\sigma} = \sum_\lambda c_\lambda \sum_{P \in \mathrm{SYT}(\lambda)} F_{[n-1]-\mathrm{Des}\,P} = \sum_\lambda c_\lambda s_{\lambda^t}$$

as desired. $\qquad\square$

In order to deal with reversals, we will need some background about the Robinson-Schensted map [Rob38, Sch61]. This is a bijection

$$\mathrm{RS} : \mathfrak{S}_n \to \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda).$$

If $\mathrm{RS}(\pi) = (P, Q)$ then we will write $P = P(\pi)$ and $Q = Q(\pi)$ and call P and Q the *P-tableau* and *Q-tableau* of π, respectively. If $\pi = \pi_1 \ldots \pi_n$ then P is constructed using an operation called *insertion* that sequentially inserts $\pi_1, \ldots, \pi_n$ to form P. After the kth insertion, a k is *placed* in Q so that one maintains $\mathrm{sh}\,P = \mathrm{sh}\,Q$ at all times. Although we are using Q in both the notation $Q(\pi)$ and $Q_n(\Pi)$ the two different concepts should be distinguishable by context and the fact that the latter has a subscript while the former does not.

We put an equivalence relation on $\mathfrak{S}_n$ by declaring that π and σ are *Knuth equivalent*, written $\pi \sim \sigma$, if $P(\pi) = P(\sigma)$. Given an SYT P, the corresponding *Knuth class* is

$$K(P) = \{\pi \mid P(\pi) = P\}.$$

Given a partition λ it will also be convenient to define an associated *Knuth aggregate* by

$$K(\lambda) = \{\pi \mid \mathrm{sh}\,P(\pi) = \lambda\} = \bigcup_{\mathrm{sh}(P)=\lambda} K(P).$$

We will need the following properties of the map RS.

Theorem 2.2. *Suppose* $\mathrm{RS}(\pi) = (P, Q)$.

(a) $\mathrm{Des}\,\pi = \mathrm{Des}\,Q$.

(b) *If* $\mathrm{sh}\,P = \lambda$ *then* λ_1 *is the length of a longest increasing subsequence of* π.

(c) $P(\pi^r) = (P(\pi))^t$.

(d) $\mathrm{RS}(\pi^{-1}) = (Q, P)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

These results have interesting implications for our quasisymmetric functions. To state them conveniently we will use the notation that for a permutation π we let $F_\pi = F_{\mathrm{Des}\,\pi}$. And for a set Π of permutations we define $F_\Pi = \sum_{\pi \in \Pi} F_\pi$.

Lemma 2.3. *Suppose* $P \in \mathrm{SYT}(\lambda)$.

(a) $F_{K(P)} = s_\lambda$.

(b) $F_{K(\lambda)} = f^\lambda s_\lambda$.

Proof. Using the fact that RS is a bijection, Theorem 1.1, and Theorem 2.2 (a) we have

$$F_{K(P)} = \sum_{\pi \in K(P)} F_{\mathrm{Des}\,\pi} = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}\,Q} = s_\lambda$$

which is (a). For (b), we use (a) to write

$$F_{K(\lambda)} = \sum_{\mathrm{sh}(P)=\lambda} F_{K(P)} = \sum_{\mathrm{sh}(P)=\lambda} s_\lambda = f^\lambda s_\lambda$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We will now prove that Proposition 2.1 continues to hold if complement is replaced by reversal.

Proposition 2.4. *If* $Q_n(\Pi)$ *is symmetric then so is* $Q_n(\Pi^r)$. *In particular, if* $Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda$ *for certain coefficients* c_λ *then*

$$Q_n(\Pi^r) = \sum_\lambda c_\lambda s_{\lambda^t}.$$

Proof. For each partition λ pick a tableau $P_\lambda \in \mathrm{SYT}(\lambda)$. Then using Lemma 2.3 (a) we have

$$\sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\mathrm{Des}\,\sigma} = Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda = \sum_\lambda c_\lambda F_{K(P_\lambda)}.$$

Combining this with Theorem 2.2 (c) yields

$$Q_n(\Pi^r) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\mathrm{Des}\,\sigma^r} = \sum_\lambda c_\lambda F_{K((P_\lambda)^t)} = \sum_\lambda c_\lambda s_{\lambda^t}$$

which is what we wished to prove. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

As an immediate corollary of Propositions 2.1 and 2.4 we get the following.

Proposition 2.5. *If $Q_n(\Pi)$ is symmetric then so is $Q_n(\Pi^{cr})$. In particular, $Q_n(\Pi^{cr}) = Q_n(\Pi)$.* □

As an application of these ideas, we will now verify the expansions in the first three and last three rows of Table 1. For the first row, we use Lemma 2.3 (b) and the fact that RS is a bijection to write

$$Q_n(\emptyset) = \sum_{\sigma \in \mathfrak{S}_n} F_\sigma = \sum_{\lambda \vdash n} F_{K(\lambda)} = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

The next two rows are special cases of the following result. In it and the sequel we eliminate the braces when writing out $Q_n(\Pi)$ for a specific Π. We will also use the notation

$$\iota_k = 12\ldots k$$

and

$$\delta_k = k\ldots 21$$

for the increasing and decreasing permutations of length k.

Proposition 2.6. *For any $k \geq 1$ we have*

$$Q_n(\iota_k) = \sum_{\lambda_1 < k} f^\lambda s_\lambda$$

and

$$Q_n(\delta_k) = \sum_{\lambda_1^t < k} f^\lambda s_\lambda.$$

Proof. By either Proposition 2.1 or Proposition 2.4, it suffices to prove the first statement. Note that $\sigma \in \mathfrak{S}_n(\iota_k)$ if and only if the longest increasing subsequence of σ has length less than k. Now using Theorem 2.2 (b), Lemma 2.3 (b) and the bijectivity of RS we obtain

$$Q_n(\iota_k) = \sum_{\sigma \in \mathfrak{S}_n(\iota_k)} F_\sigma = \sum_{\lambda_1 < k} F_{K(\lambda)} = \sum_{\lambda_1 < k} f^\lambda s_\lambda$$

which is the desired result. □

As far as the last three rows of Table 1, the reader will find it easy to prove the following result so the proof is omitted.

Proposition 2.7. *For $\Pi = \mathfrak{S}_k - \{\iota_k, \delta_k\}$ we have*

$$Q_n(\Pi) = \begin{cases} \displaystyle\sum_{\lambda \vdash n} f^\lambda s_\lambda & \text{for } n < k, \\ s_n + s_{1^n} & \text{for } n \geq k. \end{cases}$$

For $\Pi = \mathfrak{S}_k - \{\delta_k\}$ *we have*

$$Q_n(\Pi) = \begin{cases} \sum_{\lambda \vdash n} f^\lambda s_\lambda & \text{for } n < k, \\ s_{1^n} & \text{for } n \geq k. \end{cases}$$

For $\Pi = \mathfrak{S}_k - \{\iota_k\}$ *we have*

$$Q_n(\Pi) = \begin{cases} \sum_{\lambda \vdash n} f^\lambda s_\lambda & \text{for } n < k, \\ s_n & \text{for } n \geq k. \end{cases} \qquad \square$$

3 Shuffles

In this section we will show that several of the entries in Table 1 can be explained using shuffles of permutations. It turns out that under very general conditions, shuffling preserves Q_n being symmetric, and perhaps Schur nonnegative as well.

If $\pi = \pi_1 \ldots \pi_m$ and $\sigma = \sigma_1 \ldots \sigma_n$ are sequences on distinct elements on disjoint alphabets then their *shuffle set* is

$$\pi \shuffle \sigma = \{\tau = \tau_1 \ldots \tau_{m+n} \mid \pi \text{ and } \sigma \text{ are subsequences of } \tau\}.$$

For example
$$13 \shuffle 42 = \{1342, 1432, 1423, 4132, 4123, 4213\}.$$

If $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$ then we define $\pi \shuffle \sigma = \pi \shuffle (\sigma + m)$ where $\sigma + m$ is the sequence obtained by adding m to each element of σ. To illustrate

$$12 \shuffle 21 = 12 \shuffle 43 = \{1243, 1423, 1432, 4123, 4132, 4312\}.$$

We shuffle sets of permutations as expected, namely

$$\Pi \shuffle \Pi' = \bigcup_{\pi \in \Pi, \pi' \in \Pi'} \pi \shuffle \pi'.$$

To prove the next result, it will be useful to have a notation for the permutations in $\mathfrak{S}_n$ that contain a pattern in Π, which will be

$$\overline{\mathfrak{S}}_n(\Pi) = \mathfrak{S}_n - \mathfrak{S}_n(\Pi).$$

We will also use s_1 as shorthand for the Schur function $s_{(1)}$.

Theorem 3.1. *For any sets of permutations* Π *and* Π' *and any* $n \geq 0$,

$$Q_n(\Pi \shuffle \Pi') = Q_n(\Pi') + \sum_{k=0}^{n-1} Q_k(\Pi)[s_1 Q_{n-k-1}(\Pi') - Q_{n-k}(\Pi')].$$

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Proof. We first show that

$$\overline{\mathfrak{S}}_n(\Pi \sqcup \Pi') = \bigcup_{k=1}^{n-1} \overline{\mathfrak{S}}_k(\Pi) \sqcup \overline{\mathfrak{S}}_{n-k}(\Pi'). \tag{2}$$

If $\tau \in \overline{\mathfrak{S}}_n(\Pi \sqcup \Pi')$ then τ contains $\pi \sqcup \pi'$ for some $\pi \in \Pi$ and $\pi' \in \Pi'$. Let τ^a and τ^b be the copies of π and π', respectively, in τ. This implies $\max \tau^a < \min \tau^b$. Then the restriction of τ to the elements of $[k]$ where $k = \max \tau^a$ is an element of $\overline{\mathfrak{S}}_k(\Pi)$, similarly restricting to $[n] - [k]$ gives an element of $\overline{\mathfrak{S}}_{n-k}(\Pi') + k$. Thus $\tau \in \overline{\mathfrak{S}}_k(\Pi) \sqcup \overline{\mathfrak{S}}_{n-k}(\Pi')$. The reverse inclusion is proven similarly.

The *Malvenuto-Reutenauer algebra* [MR95], MR, is the set of formal $\mathbb{Q}$-linear combinations of permutations with product given by shuffle. The map $\Phi : \mathrm{MR} \to \mathrm{QSym}$ given by $\pi \mapsto F_\pi$ is a homomorphism. Define $A_k = \overline{\mathfrak{S}}_k(\Pi) \sqcup \overline{\mathfrak{S}}_{n-k}(\Pi')$. Applying Φ to both sides of the summation identity implied by (2) and then using the Principle of Inclusion-Exclusion gives

$$\Phi(\overline{\mathfrak{S}}_n(\Pi \sqcup \Pi')) = \sum_{p=1}^{n-1}(-1)^{p-1} \sum_{1 \le k_1 < \cdots < k_p \le n-1} \Phi(A_{k_1} \cap \cdots \cap A_{k_p}). \tag{3}$$

A proof similar to the one for (2) shows that if $k < \ell$ then

$$A_k \cap A_\ell = \overline{\mathfrak{S}}_k(\Pi) \sqcup \mathfrak{S}_{\ell-k} \sqcup \overline{\mathfrak{S}}_{n-\ell}(\Pi'). \tag{4}$$

It follows that if $k_1 < \cdots < k_p$, then $A_{k_1} \cap \cdots \cap A_{k_p} = A_{k_1} \cap A_{k_p}$. Applying this observation to (3) and then using the fact that $\sum_k (-1)^k \binom{n}{k} = \delta_{n,0}$ gives

$$\Phi(\overline{\mathfrak{S}}_n(\Pi \sqcup \Pi')) = \sum_{k=1}^{n-1}\Phi(A_k) + \sum_{1\le k<\ell\le n-1}\Phi(A_k \cap A_\ell) \sum_{p=2}^{n-1}(-1)^{p-1}\binom{\ell-k-1}{p-2}$$

$$= \sum_{k=1}^{n-1}\Phi(A_k) - \sum_{k=1}^{n-2}\Phi(A_k \cap A_{k+1}).$$

Now using the definition of A_k, equation (4), and the fact that Φ is a homomorphism yields

$$\Phi(\overline{\mathfrak{S}}_n(\Pi \sqcup \Pi')) = \sum_{k=1}^{n-1}\Phi(\overline{\mathfrak{S}}_k(\Pi) \sqcup \overline{\mathfrak{S}}_{n-k}(\Pi')) - \sum_{k=1}^{n-2}\Phi(\overline{\mathfrak{S}}_k(\Pi) \sqcup \mathfrak{S}_1 \sqcup \overline{\mathfrak{S}}_{n-k-1}(\Pi'))$$

$$= \sum_{k=1}^{n-1}\Phi(\overline{\mathfrak{S}}_k(\Pi))\Phi(\overline{\mathfrak{S}}_{n-k}(\Pi')) - \sum_{k=1}^{n-2}\Phi(\overline{\mathfrak{S}}_k(\Pi))s_1\Phi(\overline{\mathfrak{S}}_{n-k-1}(\Pi')).$$

Writing $\Phi(\overline{\mathfrak{S}}_n(\Pi)) = \Phi(\mathfrak{S}_n) - Q_n(\Pi) = s_1^n - Q_n(\Pi)$, expanding the binomials, and rearranging the terms left after cancellation gives the theorem. $\qquad\square$

We note that this theorem takes a nice form when expressed in terms of generating functions. In particular, if one lets

$$Q(\Pi) = \sum_{n=0}^{\infty} Q_n(\Pi)t^n$$

then the previous result becomes the following.

Corollary 3.2. *For any sets of permutations Π and Π',*

$$Q(\Pi \sqcup \Pi') = Q(\Pi) + Q(\Pi') + (ts_1 - 1)Q(\Pi)Q(\Pi').$$ □

We also have the following immediate corollary of Theorem 3.1

Corollary 3.3. *For any sets of permutations Π and Π', if $Q_n(\Pi)$ and $Q_n(\Pi')$ are symmetric for all n then then same is true of $Q_n(\Pi \sqcup \Pi')$.* □

Unfortunately, the theorem does not show that Schur nonnegativity is preserved. However computer evidence supports this conjecture.

Conjecture 3.4. *For any sets of permutations Π and Π', if $Q_n(\Pi)$ and $Q_n(\Pi')$ are Schur nonnegative for all n then then same is true of $Q_n(\Pi \sqcup \Pi')$.*

Although we can not do so in general, we can still derive Schur nonnegativity under certain circumstances.

Lemma 3.5. *Suppose that, for all $n \geq 0$*

$$G_n = \sum_{\lambda \vdash n} c_\lambda s_\lambda$$

for certain constants c_λ. Define, for all $n \geq 1$, a symmetric function G'_n and constants d_λ by

$$G'_n = s_1 G_{n-1} - G_n = \sum_{\lambda \vdash n} d_\lambda s_\lambda.$$

(a) We have

$$d_\lambda = \left(\sum_{\lambda^-} c_{\lambda^-} \right) - c_\lambda$$

where λ^- ranges over all diagrams obtained by removing a box from the diagram of λ

(b) If $c_\lambda \leq \sum_{\lambda^-} c_{\lambda^-}$ for all $\lambda \vdash n$ then G'_n is Schur-nonnegative.

Proof. Clearly (ii) follows from (i). And (i) itself follows easily from the Pieri rule for multiplying Schur functions. □

As an application, we will consider shuffles with ι_k, δ_k, and $\mathfrak{S}_k$.

Corollary 3.6. *If $Q_n(\Pi)$ is Schur-nonnegative for all $n \geq 0$ then so are the following: $Q_n(\Pi \sqcup \iota_k)$, $Q_n(\Pi \sqcup \delta_k)$, and $Q_n(\Pi \sqcup S_k)$.*

Proof. We will only prove first statement as the others are similar. Recall that the Littlewood-Richardson rule expresses the product of two Schur functions as a non-negative linear combination of Schur functions. Then in view of Theorem 3.1, it suffices to show that condition (ii) of Lemma 3.5 is satisfied where $G_n = Q_n(\iota_k)$. By Proposition 2.6, if $\lambda_1 \geq k$ then $c_\lambda = 0$ and so the inequality is immediate. On the other hand, if $\lambda_1 < k$ then the same is true of all λ^-. In this case

$$c_\lambda = f^\lambda = \sum_{\lambda^-} f^{\lambda^-} = \sum_{\lambda^-} c_{\lambda^-}$$

and we are done. □

Now we can verify another four entries in Table 1.

Theorem 3.7. *For $n \geq 2$ we have*

$$Q_n(123, 132, 312) = Q_n(123, 213, 231) = s_{1^n} + s_{2,1^{n-2}}$$

and

$$Q_n(132, 312, 321) = Q_n(213, 231, 321) = s_n + s_{n-1,1}.$$

Proof. By Propositions 2.1 and 2.4, we only need to prove the statement for one of the four sets of permutations. Consider $\{123, 132, 312\} = \{12\} \sqcup \{1\}$ and suppose $n \geq 1$. We have $Q_n(1) = \delta_{n,1}$ where we are using the Kronecker delta. Then for any Π, Theorem 3.1 gives $Q_n(\Pi \sqcup 1) = s_1 Q_{n-1}(\Pi)$. In particular, using Proposition 2.6 and the Pieri rule

$$Q_n(12 \sqcup 1) = s_1 Q_{n-1}(12) = s_1 s_{1^{n-1}} = s_{1^n} + s_{2,1^{n-2}}$$

as desired. □

4   Partial shuffles

We now wish to study and generalize $Q_n(132, 213)$ (and, by symmetry, $Q_n(231, 312)$). We will do this using a new concept that we call a partial shuffle. First, however, we will derive the quasisymmetric function for $\{132, 213\}$ itself.

There is a well-known characterization of the permutations in $\mathfrak{S}_n(132, 213)$; see, for example, the paper of Simion and Schmidt [SS85]. A permutation in $\mathfrak{S}_n$ is *reverse layered* if it is of the form

$$\pi = a, a + 1, \ldots, n, b, b + 1, \ldots, a - 1, c, c + 1, \ldots, b - 1, \ldots \tag{5}$$

For certain $a > b > c > \cdots > 0$. We are using the term "reverse" because in this case π^r is what is usually called a *layered permutation*. Also, let

$$\mathcal{H}_n = \{\lambda \vdash n \mid \lambda \text{ is a hook}\}$$

and $\mathrm{SYT}(\mathcal{H}_n)$ be the set of tableaux P so that $\mathrm{sh}(P) \in \mathcal{H}_n$.

Proposition 4.1. *We have*

$$Q_n(132, 213) = Q_n(231, 312) = \sum_{\lambda \in \mathcal{H}_n} s_\lambda$$

Proof. By symmetry, it suffices to prove that $Q_n(132, 213)$ is equal to the sum. From the description (5) it is clear that the map $\pi \mapsto \mathrm{Des}\,\pi$ gives a bijection $\mathfrak{S}_n(132, 213) \to 2^{[n-1]}$. Thus

$$Q_n(132, 213) = \sum_{S \subseteq [n-1]} F_S.$$

We will be done by Lemma 2.3 (a) if we can show that the map $P \mapsto \mathrm{Des}\,P$ is a bijection $\mathrm{SYT}(\mathcal{H}_n) \to 2^{[n-1]}$. But is is easy to see that this map has an inverse. In particular, given $S \subseteq [n-1]$ we construct the hook tableau whose first column has elements $\{1\} \cup (S+1)$ where $S+1 \subset [n]$ is obtained by adding one to each element of S. $\square$

To try and generalize the previous result, note that $\{132, 213\} = (13 \sqcup 2) - \{123\}$. We will put a hat on an element of a permutation to indicate the sequence obtained by removing that element. For example, $3\hat{2}41 = 341$. Now define the *partial shuffle*

$$(12\ldots \widehat{n-1}n) \,\overline{\sqcup}\, (n-1) = [(12\ldots \widehat{n-1}n) \sqcup (n-1)] - \{\iota_n\}.$$

To illustrate $13\,\overline{\sqcup}\,2 = \{132, 213\}$ and $124\,\overline{\sqcup}\,3 = \{1243, 1324, 3124\}$.

For the generalization of hook diagrams, let (i, j) denote the box of a Young diagram in row i and column j. And if P is a Young tableau then $P_{i,j}$ will be the entry of P in box (i, j). We will consider the set of of *fattened hooks*

$$\mathcal{H}_{n,j} = \{\lambda \vdash n \mid (2, j) \notin \lambda\}.$$

Note $\mathcal{H}_{n,2} = \mathcal{H}_n$. The following conjecture has Proposition 4.1 as the special case $j = 3$.

Conjecture 4.2. *For $j \geq 3$ we have*

$$Q_n((12\ldots \widehat{j-1}j) \,\overline{\sqcup}\, (j-1)) = \sum_{\lambda \in \mathcal{H}_{n,j-1}} f^{\bar{\lambda}} s_\lambda$$

where $\bar{\lambda}$ is λ with λ_1 replaced by $\min\{\lambda_1, j-2\}$.

While we have not been able to prove this conjecture, we have been able to make some progress on the case $j = 4$. It follows from Theorem 2.2 (b) and (c) that if $\pi \in \mathrm{Av}(\delta_m)$ then $P(\pi)$ has less than m rows. In general it is *not* true that if $Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda$ then we have $Q_n(\Pi \cup \{\delta_m\})$ is the same sum restricted to λ with $\ell(\lambda) < m$. As an example from Theorem 1.2, $Q_n(231, 312)$ is Schur non-negative but $Q_n(231, 312, 321)$ is not even symmetric for all n. However, this property seems to be enjoyed in the context of partial shuffles.

Conjecture 4.3. *For $j \geq 3$ and $m \geq 2$ we have*

$$Q_n((12\ldots \widehat{j-1}j) \,\overline{\sqcup}\, (j-1) \cup \{\delta_m\}) = \sum_\lambda f^{\bar{\lambda}} s_\lambda$$

where the sum is over all $\lambda \in \mathcal{H}_{n,j-1}$ such that $\ell(\lambda) < m$, and $\bar{\lambda}$ is λ with λ_1 replaced by $\min\{\lambda_1, j-2\}$.

Note that Conjecture 4.3 implies Conjecture 4.2 by letting $m \to \infty$.

We can give a proof of the case $j = 3$ of Conjecture 4.3. In fact, a slight modification of the proof of Proposition 4.1 will work since the bijections used there can be restricted to layered permutations with fewer than m layers and hook tableaux with fewer than m rows, both mapping onto the subsets of $[n-1]$ with fewer than m elements. This proves the following.

Proposition 4.4. *For $m \geq 2$ we have*

$$Q_n(132, 213, \delta_m) = \sum_\lambda f^{\bar{\lambda}} s_\lambda$$

where the sum is over all $\lambda \in \mathcal{H}_n$ such that $\ell(\lambda) < m$, and $\bar{\lambda}$ is λ with λ_1 replaced by 1. In particular

$$Q_n(132, 213, 321) = s_n + s_{n-1,1}$$

and by symmetry

$$Q_n(123, 231, 312) = s_{1^n} + s_{2,1^{n-2}}. \qquad \square$$

We will now prove Conjecture 4.3 in the special case $j = m = 4$, that is, we will consider

$$\Pi = \{1243, 1324, 3124, 4321\}. \tag{6}$$

Given $\sigma \in \mathfrak{S}_n$ we will write its descent set in increasing order $\operatorname{Des}\sigma = \{d_1 < d_2 < \cdots < d_\ell\}$. We also let $d_0 = 0$ and $d_{\ell+1} = n$ when convenient. Let $\alpha(\sigma)$ denote the corresponding *descent composition* so that $\alpha_i = d_i - d_{i-1}$ for $1 \leq i \leq \ell + 1$. The *increasing run* of σ corresponding to α_i is the factor of σ consisting of the elements with indices between $d_{i-1} + 1$ and d_i inclusive. For example, if $\sigma = 561342$ then $\operatorname{Des}\sigma = \{2, 5\}$, $\alpha(\sigma) = (2, 3, 1)$ and the increasing runs of σ are 56, 134, and 2. We will need the following description of the increasing runs of a σ avoiding 1243 and 3124. It is analogous to the reverse layered description of $\sigma \in \operatorname{Av}_n(132, 213)$, which is equivalent to the increasing runs of σ being consecutive integers.

Lemma 4.5. *If σ avoids 1243 and 3124 then every increasing run of σ of length at least two has the form*

$$a, b, b+1, b+2, \ldots, b+k$$

for some $a < b$ and some $k \geq 0$.

Proof. The result is trivial if the the length of the run is two, so assume it has length at least three. Let d be the maximum (and hence, last) element of the run. Suppose, towards a contradiction, that not every integer in the interval $[b, d]$ appears in this run. In particular, suppose that there is c with $b < c < d$ that is not in the run. Now either c comes before the run so that σ contains the subsequence $cabd$, or after the run so that σ contains the subsequence $abdc$. This gives a contradiction in that σ contains either a copy of 3124 or 1243. $\qquad \square$

We now define two maps that in certain cases, because of the lemma, will be inverses. The *contraction at index j* of $\sigma \in \mathfrak{S}_n$ is the permutation $c_j(\sigma) \in \mathfrak{S}_{n-1}$ obtained by removing σ_j and then standardizing what remains. The *expansion at index j* is the permutation $e_j(\sigma) \in \mathfrak{S}_{n+1}$ obtained by increasing all the elements of σ greater than σ_j by one and then inserting $\sigma_j + 1$ directly after σ_j in the result. Continuing our example above with $\sigma = 561342$, we have $c_5(561342) = 45132$ and also $e_4(45132) = 561342 = \sigma$. To state the next result it will be convenient to have the notation.

$$A(\alpha, n) = \{\sigma \in \mathfrak{S}_n(1243, 1324, 3124, 4321) \mid \alpha(\sigma) = \alpha\}.$$

Proposition 4.6. *Let $\alpha \models n$ have a part $\alpha_i \geq 3$ and let $j = \alpha_1 + \alpha_2 + \cdots + \alpha_i$. Also let $\alpha' \models n - 1$ be the composition obtained from α by replacing α_i by $\alpha_i - 1$. Then contraction and expansion restrict to maps $c_j : A(\alpha, n) \to A(\alpha', n - 1)$ and $e_{j-1} : A(\alpha', n - 1) \to A(\alpha, n)$ that are inverses of each other.*

Proof. We first show that c_j is well defined. Suppose $\sigma \in A(\alpha, n)$. Since $\alpha_i \geq 3$ we have, by Lemma 4.5, that $\sigma_j = \sigma_{j-1} + 1$. It follows that removing σ_j and standardizing will produce a descent at position $j - 1$. And since $c_j(\sigma)$ comes from standardizing a subpermutation of σ, it must still avoid the permutations in question. Therefore $c_j(\sigma) \in A(\alpha', n - 1)$ as desired.

The fact that $e_{j-1}(\tau) = \sigma$ has the correct descent composition is similar to what was done for $c_j(\sigma)$. To show that σ still avoids the four patterns, assume to the contrary that it contains a copy κ of one of them. Then κ must contain both σ_{j-1} and $\sigma_j = \sigma_{j-1} + 1$ because, if not, then κ is also in τ (possibly obtained by replacing σ_j by τ_{j-1}). Since $\sigma_j = \sigma_{j-1} + 1$ these two elements of κ must correspond to consecutive, increasing elements in the pattern, so the only option is that they become the one and two in either 1243 or 3124. But $\alpha_i \geq 3$ so that σ_{j-1} is not the first element of its run. Then replacing $\sigma_{j-1}\sigma_j$ by $\tau_{j-2}\tau_{j-1}$ gives a copy of the forbidden pattern in τ, a contradiction.

We now show that the two maps are inverses. For any permutation τ we have $c_j e_{j-1}(\tau) = \tau$. On the other hand, if $\sigma \in A(\alpha, n)$ then, again using the fact that $\sigma_j = \sigma_{j-1} + 1$, we must have $e_{j-1}c_j(\sigma) = \sigma$ because the element deleted by c_j equals the element inserted by e_{j-1}. $\square$

As for partitions, the number of parts of a composition α will be written $\ell(\alpha)$ and called the *length* of α.

Corollary 4.7. *If $\sigma \in \mathfrak{S}_n(1243, 1324, 3124, 4321)$ then $\ell(\alpha(\sigma)) \leq 5$.*

Proof. Suppose, to the contrary, that $\ell(\alpha(\sigma)) \geq 6$. Removing all but the first 6 increasing runs of σ and standardizing we can assume $\ell(\alpha(\sigma)) = 6$. Now applying the bijections c_j from the previous proposition for various values of j, we obtain a $\tau \in \mathfrak{S}_m(1243, 1324, 3124, 4321)$ that has no increasing run of length greater than 2. It follows that $m \leq 12$. But we have verified by computer that no such τ exists. $\square$

We now have all the tools in place to compute $Q_n(\Pi)$ where Π is given by (6) and just need one more definition. For compositions we will use the partial order $\alpha \leq \beta$ if $\ell(\alpha) = \ell(\beta)$ and $\alpha_i \leq \beta_i$ for all i. Note that we consider compositions of different lengths to be incomparable.

Theorem 4.8. *Let* $\Pi = \{1243, 1324, 3124, 4321\}$. *Then*

$$Q_n(\Pi) = \sum_\lambda f^{\overline{\lambda}} s_\lambda \tag{7}$$

where the sum is over all $\lambda \in \mathcal{H}_{n,3}$ *with* $\ell(\lambda) \leq 3$, *and* $\overline{\lambda}$ *is* λ *with* λ_1 *replaced by* $\min\{\lambda_1, 2\}$. *In particular, for* $n \geq 6$ *we have*

$$Q_n(\Pi) = s_n + 2s_{n-1,1} + 2s_{n-2,2} + 3s_{n-2,1,1} + 5s_{n-3,2,1} + 5s_{n-4,2,2}. \tag{8}$$

Proof. The "in particular" follows from equation (7) and the fact that when $n \geq 6$ we always have λ_1 replaced by 2 in $\overline{\lambda}$.

We will prove (7) by induction on n, where we have verified it by computer for $n \leq 10$. Consider any nonempty set $A(\alpha, n)$ for $n \geq 11$. By Corollary 4.7, the fact that this set is nonempty forces there to be a part $\alpha_i \geq 3$. Applying the contraction bijections from Proposition 4.6, we see that there is a $\beta \models 10$ such that $\#A(\beta, 10) = \#A(\alpha, n)$.

Now consider the sum on the right-hand side of (7), which, since $n \geq 11$, reduces to (8). Using Theorem 1.1, we see that we want to show that $Q_n(\Pi)$ equals

$$F_n + 2 \left[\sum_{\alpha \geq (1,1)} F_\alpha + \sum_{\alpha \geq (2,2)} F_\alpha + \sum_{\alpha \geq (1,2,1)} F_\alpha \right] + 3 \sum_{\alpha \geq (1,1,1)} F_\alpha$$

$$+ 5 \left[\sum_{\alpha \geq (2,2,1)} F_\alpha + \sum_{\alpha \geq (2,2)} F_{1\alpha} + \sum_{\alpha \geq (2,1,2)} F_\alpha + \sum_{\alpha \geq (1,2,1,1)} F_\alpha + \sum_{\alpha \geq (1,1,2,1)} F_\alpha + \sum_{\alpha \geq (2,2,2)} F_\alpha \right.$$

$$\left. + \sum_{\alpha \geq (1,2,1,2)} F_\alpha + \sum_{\alpha \geq (2,2,1)} F_{1\alpha} + \sum_{\alpha \geq (2,1,2,1)} F_\alpha + \sum_{\alpha \geq (2,2,2,1)} F_\alpha + \sum_{\alpha \geq (1,2,1,2,1)} F_\alpha + \sum_{\alpha \geq (1,1,2,1,1)} F_\alpha \right]$$

where 1α is the composition obtained by concatenating (1) and α, and all subscripts of the F's are compositions of n. To finish the induction, it suffices to show that the sums of fundamentals to which α contributes are the same as the ones to which β contributes where α and β are as in the previous paragraph. But all of the lower bounds in the sums are compositions with only ones and twos. And applying c_j reduces a part of size at least three to a part of size at least two and leaves all other parts the same. Therefore this map does respect the expansion into fundamentals above and we are done. □

Note that (8) shows that the Schur expansion for this set of patterns stabilizes as $n \to \infty$. We will have more to say about stability in the next and last sections.

5 Knuth classes

The final entries from Table 1 that need to be explained are $\{132, 312\}$ and $\{213, 231\}$. The reader will recognize these as Knuth classes $K(P)$ for P equal to

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \tag{9}$$

and

$$\begin{array}{|c|c|}\hline 1 & 3 \\\hline 2 \\\cline{1-1}\end{array}\ ,$$ (10)

respectively. The main result of this section will be a characterization of the P for which $\mathfrak{S}_n(K(P))$ is a union of Knuth classes for all n. For such P it is clear that $Q_n(K(P))$ will be Schur nonnegative. We will first deal with the special case when P is one of the two tableaux above.

We will concentrate on $\{132, 312\}$ since the other Π is just its reversal. It follows easily from the inductive description of the $\sigma \in \mathfrak{S}_n(132, 312)$ in [SS85] that they are exactly the permutations such that for all $j \geq 2$ we have

$$\sigma_j = \begin{cases} (\min_{i<j}\{\sigma_i\}) - 1, & \text{or} \\ (\max_{i<j}\{\sigma_i\}) + 1. \end{cases}$$

Equivalently, for all $j \geq 1$ the set $\{\sigma_1, \sigma_2, \ldots, \sigma_j\}$ is an interval of integers. In the representation-theoretic proof of [ER14, Proposition 7.3] (see Section 10.2), Elizalde and Roichman mention a result closely related to the following.

Proposition 5.1. *For $n \geq 1$ we have*

$$Q_n(132, 312) = Q_n(213, 231) = \sum_{\lambda} s_\lambda$$

where the sum is over all hook diagrams λ with n boxes.

Proof. By reversal, it suffices to prove that $Q_n(132, 312)$ is equal to the sum. As in the proof of Proposition 4.1, it suffices to show that Des restricts to a bijection $\mathfrak{S}_n(132, 312) \to 2^{[n-1]}$. But given $J \subseteq [n-1]$ there is a unique σ avoiding $\{132, 312\}$ with that descent set, which can be constructed as follows. Arrange the integers $1, 2, \ldots, \#J$ in a decreasing sequence in the positions of $J+1$ (the set obtained by adding one to every element of J) of σ. Then arrange the integers $\#J + 1, \#J + 2, \ldots, n$ as an increasing sequence in the remaining positions. It is easy to check that the permutation thus constructed has $\mathrm{Des}\,\sigma = J$. $\qquad\square$

We now consider general Knuth classes $K(P)$. We first need to recall some general facts about these sets. Suppose we have positive integers $a < b < c$ and a permutation σ having a factor (subsequence of consecutive elements) acb or cab. Then we can perform a *Knuth move* on σ by exchanging one factor for the other. Alternatively, a Knuth move can exchange factors of the form bac and bca. Two permutations are *Knuth equivalent* if one can be transformed into the other by a sequence of Knuth moves.

Theorem 5.2. *If $P(\sigma) = P$ then $K(P)$ is the set of permutations Knuth equivalent to σ.* $\quad\square$

Call Π *pattern-Knuth closed* if $\mathfrak{S}_n(\Pi)$ is a union of Knuth classes for all n. Equivalently, the complement $\overline{\mathfrak{S}}_n(\Pi)$ is a union of Knuth classes for all n. The following lemma is easy to prove directly from the definitions so its demonstration is omitted.

Lemma 5.3. *If Π is pattern-Knuth closed then so are Π^c, Π^r, and Π^{rc}.* $\qquad\square$

There is a sense in which this property is stable.

Lemma 5.4. *The set Π is pattern-Knuth closed if and only if $\mathfrak{S}_n(\Pi)$ is a union of Knuth classes for $n \leq M + 1$ where M is the maximum length of a permutation in Π.*

Proof. The forward direction is obvious so we will prove the converse. It will be more convenient to show that $\overline{\mathfrak{S}}_n(\Pi)$ is a union of Knuth classes for $n > M+1$. Suppose $\sigma \in \overline{\mathfrak{S}}_n(\Pi)$ contains a copy κ of some $\pi \in \Pi$. By Theorem 5.2, it suffices to show that the result σ' of performing a Knuth move on σ will still be in $\overline{\mathfrak{S}}_n(\Pi)$. We will do this for a factor acb where $a < b < c$ as the other possible Knuth moves can be handled similarly. There are three cases.

If κ does not contain both a, c then κ is still a subsequence of σ' and we are done. If κ contains all three elements, then σ' contains κ', which is formed by replacing acb by cab in κ. Since our hypothesis implies that Π is a union of Knuth classes, κ' is a copy of an element of Π and we are again finished. The final case is when a, c are in κ but b is not. Let π' be the standardization of the subsequence of σ containing κ and b. Then $\pi' \in \overline{\mathfrak{S}}_n(\Pi)$ for some $n \leq M+1$. It follows from the hypothesis in this direction that the permutation π'' obtained by switching the elements corresponding to a, c in π' still contains an element of Π. Since σ' contains a copy of π'' this finishes the proof. $\square$

We now show that being pattern-Knuth closed is preserved by shuffling.

Lemma 5.5. *If $A \subseteq \mathfrak{S}_m$ and $B \subseteq \mathfrak{S}_n$ are unions of Knuth classes then so is $A \sqcup\!\sqcup B$.*

Proof. We will show that if $\alpha \in A$, $\beta \in B$ and $\sigma \in \alpha \sqcup\!\sqcup \beta$ contains the factor acb where $a < b < c$, then replacing this factor by cab results in $\sigma' \in A \sqcup\!\sqcup B$. The proof for the other Knuth moves is similar. There are three cases.

If $a \leq m < c$ then α and $\beta + m$ are still subwords of σ' and so we are done. If $c \leq m$ then acb is a factor of α. But A is a union of Knuth classes so replacing this factor by cab gives $\alpha' \in A$. Thus $\sigma' \in \alpha' \sqcup\!\sqcup \beta \subseteq A \sqcup\!\sqcup B$. The only other possibility is $a > m$ in which case a similar argument to the one just given with A replaced by B completes the proof. $\square$

Combining this lemma and equation (2) we immediately see that Conjecture 3.4 is true in the case when Π and Π' are pattern-Knuth closed. In fact, we only need to assume that one of Π and Π' is pattern-Knuth closed.

Proposition 5.6. *If Π and Π' are pattern-Knuth closed then so is $\Pi \sqcup\!\sqcup \Pi'$. If $Q_n(\Pi)$ is Schur nonnegative for all n and Π' is pattern-Knuth closed, then $Q_n(\Pi \sqcup\!\sqcup \Pi')$ is Schur nonnegative for all n.* $\square$

Proof. The first sentence follows from Lemma 5.5 and equation (2). Under the hypotheses of the second sentence, Theorem 3.1 implies that $Q_n(\Pi \sqcup\!\sqcup \Pi')$ will be Schur nonnegative for all n if the same is true of $s_1 Q_{n-1}(\Pi') - Q_n(\Pi')$. Recalling the homomorphism $\Phi : \mathrm{MR} \to \mathrm{QSym}$ sending $w \mapsto F_w$ from the proof of Theorem 3.1, we have

$$s_1 Q_{n-1}(\Pi') - Q_n(\Pi') = \Phi\left(\sum_{w \in A} w\right)$$

where $A = (\mathfrak{S}_{n-1}(\Pi') \sqcup\!\sqcup \iota_1) \setminus \mathfrak{S}_n(\Pi')$. But A is a union of Knuth classes by Lemma 5.5 and the assumption that Π' is pattern-Knuth closed. $\square$

We need one more result before we prove the main theorem of this section. For $J \subseteq [n-1]$ we let
$$D_J = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des}\pi = J\}.$$
Suppose that $\operatorname{RS}(\pi) = (P,Q)$. Then, by Theorem 2.2 (a), we have $\pi \in D_J$ if and only if $\operatorname{Des}Q = J$. As with other operations, we apply the inverse operator to a set of permutations by applying it to each individual permutation. It now follows from what we have just said and Theorem 2.2 (d) that $\pi \in D_J^{-1}$ if and only if $\operatorname{Des}P = J$. Then D_J^{-1} is a union of Knuth classes. Also, it follows easily from the definitions that $\operatorname{Des}(\pi^{-1})$ is the set of all i such that $i+1$ appears to the left of i in π. These two observations are important for the proof of the following result.

Lemma 5.7. *For any J, the set D_J^{-1} is pattern Knuth closed.* $\qquad\square$

Proof. The proof is very similar to that of Lemma 5.4. We only need to take some care with the last case for which we use the same notation as in that demonstration. Suppose first that the elements of π corresponding to a, c in κ are not consecutive in value. From the discussion before the lemma, switching them will result in κ' whose standardization has the same inverse descent set as κ, namely J. Thus $\sigma' \in \overline{\mathfrak{S}}_n(D_J^{-1})$. Now suppose that a and c standardize respectively to i and $i+1$ in π. It follows that the other elements of κ are all less than a or greater than c. Now let κ' be κ with c replaced by b. Since $a < b < c$ we have that κ' also standardizes to π, and κ' is a subsequence of σ', finishing this subcase. $\qquad\square$

To state our principal result, we need one last set of definitions. The *row superstandard* Young tableau of shape λ is the SYT obtained by filling the first row with $1, 2 \ldots, \lambda_1$; the next row with $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$; and so on. Note that the tableau in (9) is row superstandard. A *column superstandard* Young tableau is defined similarly except that one fills the columns from left to right; see (10). A tableau of either type is just called *superstandard*

Theorem 5.8. *The class $K(P)$ is pattern-Knuth closed if and only if P is a superstandard tableau of hook shape.*

We will prove this in a sequence of propositions which deal with the various cases involved. Note that because of Theorem 2.2 (c) and Lemma 5.3 we have $K(P)$ is pattern-Knuth closed if and only if $K(P^t)$ is. Therefore we only need to prove closure, or lack thereof, for one of P or P^t. We also assume throughout that the shape of P is $\lambda \vdash n$.

Proposition 5.9. *If P is superstandard of hook shape then $K(P)$ is pattern-Knuth closed.*

Proof. Suppose $\lambda = (n-k, 1^k)$. By the remarks just before this proposition it suffices to prove this when P is column superstandard. It is easy to see that in this case $K(P) = D_J^{-1}$ where $J = [k]$. The result then follows from Lemma 5.7. $\qquad\square$

It will be convenient in our proofs to work with permutations and SYT having elements that are rational numbers, not just integers. To this end, given an integer a we let $a^+ = a + 1/2$ and $a^- = a - 1/2$. It will be important when comparing such permutations and

tableaux that we always standardize them to have entries $[n]$ for some n. We will also need the *column reading word* of an SYT P, which is the permutation $\rho(P)$ obtained by recording the elements in each column of P read bottom to top and then concatenating the sequences for the columns left to right. It is easy to see that the insertion tableau of $\rho(P)$ is P.

Proposition 5.10. *If P is of hook shape but not superstandard then $K(P)$ is not pattern-Knuth closed.*

Proof. Our strategy, as suggested by Lemma 5.4, will be to find an element $\sigma \in \overline{\mathfrak{S}}_{n+1}(K(P))$ to which a Knuth move can be applied creating a permutation σ' having no subsequence κ with $\operatorname{std} P(\kappa) = P$. By transposition if necessary, we can assume that n is in the first column of P. Since P is not superstandard, there is some $a > 1$ in its first column such that $a+1$ is in the first row of P. Let a be the largest such element. Let $b > a+1$ be the next element in P's first column so that we have the following situation

$$P = \begin{array}{|c|c|c|c|}
\hline
1 & \cdots & a{+}1 & \cdots \\
\hline
\vdots \\
\cline{1-1}
a \\
\cline{1-1}
b \\
\cline{1-1}
\vdots \\
\cline{1-1}
n \\
\cline{1-1}
\end{array}$$

with $b+1$ being in the first column if $b < n$.

Let σ be the permutation obtained by placing a^+ just before $b \in \rho(P)$ so that

$$\sigma = n, \dots, a^+, b, a, \dots$$

which by construction is an element of $\overline{\mathfrak{S}}_{n+1}(\Pi)$. Now exchange a and b to obtain

$$\sigma' = n, \dots, a^+, a, b, \dots$$

Knuth equivalent to σ. Let κ be any subsequence of σ' with $|\kappa| = n$ and suppose k is the element removed to form κ. If $k = b$ then the first column of $P(\kappa)$ contains a^+, which standardizes to $a+1$ so that $\operatorname{std} P(\kappa) \neq P$. If $k \neq b$ then when b is inserted it will enter the tableau in box $(1,2)$ since either a or a^+ will be in box $(1,1)$ at that point. When $P(\kappa)$ is standardized, b will either stay the same or be replaced by $b+1$. But both those elements need to be in the first column, so at some point b will be bumped out of the first row. But then it will enter the second row in box $(2,2)$ because at least two of $1, a, a^+$ are in κ so that the element in box $(2,1)$ when b is bumped will be a or smaller. Then there will be a $(2,2)$ entry in $P(\kappa)$ forcing its shape not to be a hook, which completes our proof. $\qquad\square$

Note that by performing transpositions, it suffices to prove Theorem 5.8 for tableaux P satisfying $P_{2,1} = 2$.

Proposition 5.11. *Let P be a standard tableau of non-hook shape with $P_{2,1} = 2$. If there exists an i with $P_{i,1} > P_{i-1,2}$ then $K(P)$ is not pattern-Knuth closed.*

Proof. We will use the strategy and notation of the proof of the previous proposition. We can write P in the following form

$$P = \begin{array}{|c|c|} \hline 1 & c \\ \hline 2 & d \\ \hline \end{array} \quad S$$

It is easy to see that the insertion tableau of $\pi = \rho(R), 2, d, 1, c, \rho(S)$ is P. We consider

$$\sigma = \rho(R), 2, d, 2^+, 1, c, \rho(S),$$

which is Knuth equivalent to

$$\sigma' = \rho(R), d, 2, 2^+, 1, c, \rho(S).$$

Remove an element k from σ' to form κ. There are two cases.

Suppose first that $k \notin S_1 \cup \{2, 2^+, c\}$ where S_1 is the first row of tableau S. It follows that $2, 2^+, c, S_1$ is an increasing subsequence of κ of length $\lambda_1 + 1$ where $\operatorname{sh} P = \lambda$. Thus, by Theorem 2.2 (b), $P(\kappa)$ has first row of length longer than λ_1 and so is not of the correct shape. If $k \in S_1 \cup \{2, 2^+, c\}$ then $k \neq 1, d$. Let i be an index with $P_{i,1} > P_{i-1,2}$ and note that $i \geq 3$ so that $P_{i-1,2} \geq d$. It follows that $P_{\ell(\lambda),1}, \ldots, P_{i,1}, P_{i-1,2}, \ldots, P_{3,2}, d, x, 1$ is a decreasing subsequence of κ where x is either 2 or 2^+. But this subsequence has length $\ell(\lambda) + 1$. If follows from Theorem 2.2 (b) and (c) that $P(\kappa)$ will have first column of length longer than $\ell(\lambda) + 1$ and so, again, will not have the right shape. This finishes the proof. $\square$

We can extend the definition of P by letting $P_{i,j} = \infty$ for $i, j \geq 1$ and (i, j) outside $\lambda = \operatorname{sh} P$. In this case, the proof of the previous proposition still goes through in the case where P's first two columns are of equal length since $P_{\ell(\lambda)+1,1} = \infty > P_{\ell(\lambda),2}$. We can now finish the proof of Theorem 5.8 with the following proposition.

Proposition 5.12. *Let P be of non-hook shape. If $P_{i+1,1} < P_{i,2}$ for all $1 \leq i \leq t$ where t is the length of P's second column, then $K(P)$ is not pattern-Knuth closed.*

Proof. We can write

$$P = \begin{array}{|c|c|} \hline R & \\ \hline a & c \\ \hline b_0 & e \\ \hline C & \\ \hline \end{array} \quad S$$

where C is a single column. Note that by the discussion just before this proposition, C must be nonempty. Let $\rho(C) = f_p \ldots f_1 d_l \ldots d_1 b_k \ldots b_1$ where

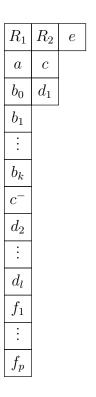$$a < b_0 < b_1 < \cdots < b_k < c < d_1 < \cdots < d_l < e < f_1 < \cdots < f_p.$$

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Figure 2: The insertion tableau for σ' in Case 1.

Note that either $k > 0$ or $l > 0$ since C must contain an element less than e to satisfy the hypothesis of this proposition. Our proof splits into two cases.
Case 1: $l > 0$. Define

$$\pi = \rho(C), b_0, e, a, c, \rho(R), \rho(S).$$

It easy to check that $P(\pi) = P$. Also let

$$\sigma = f_p, \ldots, f_1, d_l, \ldots, d_2, c^-, d_1, b_k, \ldots, b_0, e, a, c, \rho(R), \rho(S)$$

which is obtained from π by adding c^-. Apply a Knuth move to obtain

$$\sigma' = f_p, \ldots, f_1, d_l, \ldots, d_2, c^-, b_k, d_1, b_{k-1}, \ldots, b_0, e, a, c, \rho(R), \rho(S)$$

Remove any h from σ' to obtain κ. We will show $P(\kappa) \neq P$.

Consider the insertion tableau of σ up to and including $\rho(R)$, which will have the form given in Figure 2 where R_1 and R_2 are the two columns of R. If h comes from the first column of this tableau then its removal will cause all the entries to shift up by one, making the first column too short. The only way to compensate for this is if the insertion of $\rho(S)$ causes e to bump into the first column in row r just below b_0. But then

$$P(\kappa)_{r,1} = e > d_1 = P(\kappa)_{r-1,2}$$

which is a contradiction. On the other hand, if h comes from the second column of Figure 2 then e will have to end up in the second column to preserve column lengths. And if h comes

from a column of S then the first two columns in the figure must remain undisturbed. In either case the first column will contain more entries in the interval $[b_0, c]$ than P does, which finishes this case.

Case 2: $l = 0$ and $k > 0$. Now we consider

$$\pi = f_p, \ldots, f_1, b_k, e, b_{k-1}, \ldots, b_0, a, c, \rho(R), \rho(S).$$

It is easily checked that since $k > 0$ we have $P(\pi) = P$. Add c^+ to obtain

$$\sigma = f_p, \ldots, f_1, c^+, b_k, e, b_{k-1}, \ldots, b_0, a, c, \rho(R), \rho(S)$$

and perform a Knuth move to yield

$$\sigma' = f_p, \ldots, f_1, c^+, e, b_k, b_{k-1}, \ldots, b_0, a, c, \rho(R), \rho(S).$$

Define h, κ, and R_1 as before with the goal of showing $P(\kappa) \neq P$.

Note that there is a decreasing subsequence of σ' of length $\ell(\lambda) + 1$ consisting of the f's, either c^+ or e, the b's, a, and $\rho(R_1)$. Therefore $h \in \{a, b_0, \ldots, b_k, f_1, \ldots, f_p\} \cup R_1$. If h is one of the f's then $P(\kappa)$ will have more elements in the interval $[a, e]$ in its first column than P. If h is any of the other possibilities, then $P(\kappa)$ will have more elements greater than c in its first column than P. Either way, we have our final contradiction. $\qquad\square$

We conclude this section with a two questions. First of all, we know from Lemma 2.3 Theorem 5.8 that when P is superstandard of hook shape then $Q_n(\Pi)$ is Schur nonnegative.

Question 5.13. *Find a combinatorial interpretation for the coefficients in the Schur expansion for $Q_n(K(P))$ when P is superstandard of hook shape.*

It is also natrual to ask about generalizing Theorem 5.8 to pairs of tableaux.

Question 5.14. *Let P, Q be standard Young tableaux. Find a necessary and sufficient condition for $K(P) \cup K(Q)$ to be pattern-Knuth closed.*

We note that if Π, Π' are both pattern-Knuth closed then so is $\Pi \cup \Pi'$ since in this case $\overline{\mathfrak{S}}_n(\Pi \cup \Pi')$ is the union of the Knuth classes of $\overline{\mathfrak{S}}_n(\Pi)$ and $\overline{\mathfrak{S}}_n(\Pi')$. This together with Theorem 5.8 shows that if P, Q are superstandard hooks then $K(P) \cup K(Q)$ is pattern-Knuth.

6 Arc permutations

Elizalde and Roichman introduced and studied arc permutations in [ER14, ER15]. They are closely related to two of the permutation classes we have been considering and so we will be able to produce a bijection between the arc permutations and permutations avoiding a certain shuffle.

A permutation $\sigma \in \mathfrak{S}_n$ is in the set of *arc permutations*, $\mathcal{A}_n$, if each prefix $\sigma_1 \sigma_2 \ldots \sigma_i$ is an interval in the integers modulo n. Equivalently, $\mathcal{A}_n = \mathfrak{S}_n(\Pi_{\mathcal{A}})$ where

$$\Pi_{\mathcal{A}} = \{1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231\}.$$

Recall that the permutations avoiding $\{132, 312\}$ were those where each prefix was an interval of integers and a similar statement is true for suffixes when avoiding $\{213, 231\}$. Define the *pin shaped* permutations to be

$$\mathcal{P}_n = \mathfrak{S}_n(132, 312) \cup \mathfrak{S}_n(213, 231) \subseteq \mathcal{A}_n.$$

(Elizalde and Roichman call these permutations *unimodal* because their rotations are unimodal sequences, but we prefer to reserve unimodal for its original meaning.) We will also be using the complement $\mathcal{Z}_n = \mathcal{A}_n - \mathcal{P}_n$.

In order to define the symmetric functions to describe $Q_n(\Pi_{\mathcal{A}})$ we will need a few more definitions. Consider the set of non-trivial hooks

$$\overline{\mathcal{H}}_n = \mathcal{H}_n - \{(n), (1^n)\}$$

as well as the diagrams one obtains from the elements of $\overline{\mathcal{H}}_n$ by adding the $(2, 2)$ box

$$\mathcal{T}_n = \{\lambda \cup (2, 2) \mid \lambda \in \overline{\mathcal{H}}_{n-1}\}.$$

Define the corresponding generating functions $\overline{H}_n = \sum_{\lambda \in \overline{\mathcal{H}}_n} s_\lambda$ and $T_n = \sum_{\lambda \in \mathcal{T}_n} s_\lambda$. The following result is a consequence of [ER14, Theorem 7.7].

Theorem 6.1. *For all $n \geq 0$ we have*

$$Q_n(\Pi_{\mathcal{A}}) = T_n + 2\overline{H}_n + s_n + s_{1^n} \qquad \square$$

There is a shuffle class with the same expansion. Let

$$\Pi_{\mathcal{S}} = \{1\} \sqcup \{132, 312\} = \{1243, 1423, 2143, 4123, 2413, 4213, 2431, 4231\}.$$

Proposition 6.2. *For all $n \geq 0$ we have*

$$Q_n(\Pi_{\mathcal{S}}) = Q_n(\Pi_{\mathcal{A}}).$$

Proof. Applying Theorem 3.1 with $\Pi = \{1\}$ and $\Pi' = \{132, 312\}$, then Proposition 5.1, and finally the Pieri formula gives

$$Q_n(\Pi_S) = Q_n(\Pi') + s_1 Q_{n-1}(\Pi') - Q_n(\Pi') = T_n + 2\overline{H}_n + s_n + s_{1^n}.$$

Comparing this with the previous result completes the proof. $\qquad \square$

Note that by reversal, complement and Proposition 4.1, the previous proposition is true if $\Pi_{\mathcal{S}}$ is replaced by either $\{1\} \sqcup \Pi$ or $\Pi \sqcup \{1\}$ for any

$$\Pi \in \{ \{132, 312\}, \{132, 213\}, \{213, 231\}, \{231, 312\} \}.$$

However, there is no dihedral symmetry relating any of these shuffles to $\Pi_{\mathcal{S}}$.

Elizalde and Roichman [ER14, Section 7.4] gave a bijective proof of Theorem 6.1. In particular, they proved the following result.

Theorem 6.3. *For all $n \geq 0$, there is an explicit bijection*

$$\phi : \mathcal{Z}_n \to \bigcup_{\lambda \in \mathcal{T}_n} \mathrm{SYT}(\lambda).$$ □

As a consequence, we have the following.

Corollary 6.4. *For all $n \geq 0$, there is an explicit bijection*

$$\psi : \mathfrak{S}_n(\Pi_{\mathcal{A}}) \to \mathfrak{S}_n(\Pi_{\mathcal{S}}).$$

Proof. Note that by their descriptions in terms of prefixes and suffixes, we have

$$\mathfrak{S}_n(132, 312) \cap \mathfrak{S}_n(213, 231) = \{\iota_n, \delta_n\}.$$

For $\sigma \in \mathfrak{S}_n(\Pi_{\mathcal{A}})$ we define $\psi(\sigma)$ as follows, using $Q(\pi)$ as the Robinson-Schensted recording tableau.

(a) If $\sigma \in \mathfrak{S}_n(132, 312)$ then let $\psi(\sigma) = \sigma$.

(b) If $\sigma \in \mathcal{Z}_n$ then let $\psi(\sigma) = \tau$ where $\tau \in \mathfrak{S}_n(\Pi_{\mathcal{S}})$ is the unique permutation with $Q(\tau) = \phi(\sigma)$.

(c) If $\sigma \in \mathfrak{S}_n(213, 231) - \{\iota_n, \delta_n\}$ then let $\psi(\sigma) = \tau$ where $\tau \in \mathfrak{S}_n(\Pi_{\mathcal{S}}) - \mathfrak{S}_n(132, 312)$ is the unique permutation with $Q(\tau) = Q(\sigma)$.

Note that (b) is well defined by Lemma 5.5 and the fact that $Q_n(\mathcal{Z}_n)$ is Schur multiplicity free. In a similar manner we see that (c) is well defined. The check that this is a bijection is now easily done. □

The construction of the map ψ is hardly as illuminating as one would hope.

Question 6.5. *Is there a direct description of a bijection between $\mathfrak{S}_n(\Pi_{\mathcal{A}})$ and $\mathfrak{S}_n(\Pi_{\mathcal{S}})$ on the level of permutations?*

7 Comments and open questions

We end with some comments and some questions that we hope the reader will be interested in answering.

Symmetry vs nonnegativity. It is possible to construct Π such that $Q_n(\Pi)$ is symmetric, but not Schur nonnegative. In particular, consider the following set of permutations where we have enclosed certain elements in parentheses for readability

$$X_n = \{\iota_n\} \cup \{2314\ldots n, \ 12\ldots(n-3)n(n-2)(n-1)\}$$
$$\cup \{2134\ldots n, \ 1324\ldots n, \ \ldots, \ 12\ldots(n-2)n(n-1)\}$$
$$\cup \{32145\ldots n, \ 14325\ldots n, \ \ldots, \ 12\ldots(n-3)n(n-1)(n-2)\}.$$

Let $\Pi = \mathfrak{S}_4 - X_4$. Then one can verify that $\mathfrak{S}_n(\Pi) = X_n$ and so

$$Q_n(\Pi) = s_n + 2s_{n-1,1} + s_{n-2,1,1} - s_{n-2,2}.$$

Stability. The following is a natural question given the results we have proved such as Lemma 5.4.

Question 7.1. *Suppose Π is a nonempty and M is the maximum length of a permutation in Π. Is there an N, a function of M, such that $Q_n(\Pi)$ symmetric for $n < N$ implies that it continues to be symmetric for $n \geq N$? What about the same question with "symmetric" replaced by "Schur nonnegative?"*

It is worth pointing out that the converse of these questions is false. In particular, let $\lambda = (3, 1, 1)$ and

$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \ .$$

Then Bloom and Sagan [personal communication] have shown that if $\Pi = K(\lambda) - K(P)$ then $Q_n(\Pi)$ is Schur nonnegative for $n \geq 7$ but not even symmetric for $n = 6$.

Knuth classes. It would be interesting to determine when the union of Knuth classes is pattern-Knuth closed, generalizing Theorem 5.8. Of course, if K and L are pattern-Knuth closed then so is $K \cup L$. In the case that one gets a pattern-Knuth closed class Π, one would also like to characterize the coefficients in the Schur expansion of $Q_n(\Pi)$. Bloom and Sagan [personal communication] have done this in the case where $\Pi = K(P)$ for a superstandard hook tableau P.

Representation theory. Adin and Roichman [AR15] have developed a way to connect certain subsets of permutations with representations of $\mathfrak{S}_n$. Given $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ and $\pi \in \mathfrak{S}_n$ the α-*decomposition of* π is the factorization $\pi = \pi^{(1)} \ldots \pi^{(k)}$ where $\#\pi^{(i)} = \alpha_i$ for all i. The α-*descent set* and α-*descent number* of π are

$$\mathrm{Des}_\alpha \pi = \bigcup_i \mathrm{Des}\, \pi_i \quad \text{and} \quad \mathrm{des}_\alpha \pi = \#\, \mathrm{Des}_\alpha \pi,$$

respectively. An integer sequence $a_1 \ldots a_p$ is *comodal* (*complement unimodal*) if, for some m, we have $a_1 > \cdots > a_m < \cdots < a_p$. (Note that Adin and Roichman call such a sequence "unimodal" but that is at variance with standard practice.) Say that $\pi \in \mathfrak{S}_n$ is α-*comodal* if each $\pi^{(i)}$ in its α-decomposition is comodal. If Π is a set of permutations then let Π_α denote the set of α-comodal permutations in Π. Finally, call $\Pi \subseteq \mathfrak{S}_n$ *fine* if there is an $\mathfrak{S}_n$-character χ such that for all $\alpha \models n$

$$\chi(\alpha) = \sum_{\pi \in \Pi_\alpha} (-1)^{\mathrm{des}_\alpha \pi}$$

where $\chi(\alpha)$ is the value of χ on the conjugacy class indexed by α.

Adin and Roichman give a number of conditions equivalent to a set of permutations $\Pi \subseteq \mathfrak{S}_n$ being fine, one of which is that $Q_n(\Pi)$ is symmetric and Schur nonnegative. They also state that the only known examples of fine permutations are

1. permutations of given Coxeter length,

2. unions of conjugacy classes,

3. unions of Knuth classes, and

4. arc permutations.

It is clear that the present work expands on this list.

Other bases. A couple of the generating functions we have calculated are Schur P-functions. For example, if $\Pi = \{132, 312\}$ then $Q_n(\Pi) = P_n$ by Proposition 5.1, and $Q(\Pi_{\mathcal{A}}) = P_{n-1,1}$ by Theorem 6.1.

Question 7.2. *For which Π is $Q_n(\Pi)$ Schur P-nonnegative for all n?*

One could also change the set of quasisymmetric functions being used in QSym_n. Let Θ_{Pk} be the peak fundamental quasisymmetric function associated with a subset $\mathrm{Pk} \subseteq [2, n-1]$. Stembridge [Ste97] developed a theory of enriched P-partitions (P here is a partially ordered set), which allowed him to prove an analogue of Theorem 1.1 where s_λ is replaced by a Schur P-function and the $F_{\mathrm{Des}\, P}$ are replaced by $\Theta_{\mathrm{Pk}(\sigma)}$ for certain permutations σ, where $\mathrm{Pk}(\sigma)$ is the peak set of σ. Then it is natural to define for a set of permutations Π

$$R_n(\Pi) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} \Theta_{\mathrm{Pk}(\sigma)}$$

and ask the question

Question 7.3. *For which Π is $R_n(\Pi)$ symmetric? In that case when is it Schur P-nonnegative?*

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