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A maj Statistic for Set Partitions

BRUCE E. SAGAN

We propose a weighting of set partitions which is analogous to the major index for permutations. The corresponding weight generating function yields the q-Stirling numbers of the second kind of Carlitz and Gould. Other interpretations of maj are given in terms of restricted growth functions, rook placements and reduced matrices. The Foata bijection interchanging inv and maj for permutations also has a version for partitions. Finally, we generalize these constructions to an analog of Rawling's $r$maj and to two new kinds of p, q-Stirling numbers.

1. THE MAJOR INDEX OF A PARTITION

Versions of the q-Stirling numbers of the second kind were first introduced by Carlitz [1, 2] and Gould [7]. Later, Milne [10] showed that those of the second kind could be viewed combinatorially as generating functions for an inversion statistic on partitions. It is well known that the q-binomial coefficients describe the distribution of two statistics on permutations: the inversion number (inv) and the major index (maj). Thus it is natural to hope for an analog of maj for partitions. The purpose of this paper is to describe such an analog and some of its properties.

The rest of this section is devoted to basic definitions. In Sections 2 and 3 we will discuss other interpretations of the major index in terms of restricted growth functions, rook placements and reduced matrices, as has been done by Milne [10], Garsia and Remmel [5], Wachs and White [13] and Leroux [8] for various versions of the inversion number. Since both inv and maj have the same distribution, there should be a bijection interchanging the two. Foata [3] gave such a map for permutations and we present the partition analog in Section 4. Next, we generalize both statistics using Rawlings' $r$maj [11]. Section 6 considers joint distributions, yielding two new kinds of p, q-Stirling numbers. Finally, we end with some comments.

Let $\underset{\sim}{n} = \{1, 2, \ldots, n\}$. The set of all partitions of $\underset{\sim}{n}$ into k disjoint subsets or blocks will be denoted $S(\underset{\sim}{n}, k)$. Thus the *ordinary Stirling numbers of the second kind* are $S(n, k) = |S(\underset{\sim}{n}, k)|$, where $|\cdot|$ denotes cardinality. The blocks of $\pi \in S(\underset{\sim}{n}, k)$ will be written as capital letters separated by slashes, while elements of the blocks will be set in lower case. Furthermore, we will always put $\pi = B_1/B_2/\cdots/B_k$ in *standard form* with

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

Let d_i be the number of elements $b \in B_i$ such that $b > \min B_{i+1}$. The *descent multiset* of π is

$$\operatorname{Des} \pi = \{\{1^{d_1}, 2^{d_2}, \ldots, (k-1)^{d_{k-1}}\}\},$$

where i^{d_i} means that i is repeated d_i times. Thus each element of B_i that is greater than the minimum of the next block contributes an i to $\operatorname{Des} \pi$. The *major index* of π is just the sum of the decents:

$$\begin{aligned}\operatorname{maj} \pi &= \sum_{i \in \operatorname{Des} \pi} i \\ &= 1d_1 + 2d_2 + \cdots + (k-1)d_{k-1}.\end{aligned}$$

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For example, if

$$\pi = 1\,3\,8\,/\,2\,/\,4\,6\,7\,/\,5\,9 \tag{1}$$

then $\mathrm{Des}\,\pi = \{\{1, 1, 3, 3\}\} = \{\{1^2, 3^2\}\}$ and $\mathrm{maj}\,\pi = 1+1+3+3 = 8$.

Now define a_i to be the number $b \in B_i$ with $b > \min B_{i-1}$ and let

$$\widehat{\mathrm{Des}}\,\pi = \{\{2^{a_2}, 3^{a_3}, \ldots, k^{a_k}\}\}$$

be the *dual descent multiset* of $\pi \in S(\underline{n}, k)$. Since π is in standard form, $a_i = |B_i|$ for $i \geq 2$. The *dual major index* is then defined to be

$$\widehat{\mathrm{maj}}\,\pi = \sum_{i \in \widehat{\mathrm{Des}}\pi} (i-1).$$

In the example above, $\widehat{\mathrm{maj}}\,\pi = 1+2+2+2+3+3 = 13$.

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ be the natural numbers and let q be an indeterminate. If $k \in \mathbf{N}$ then the *q-analog of k* is

$$[k] = 1 + q + q^2 + \cdots + q^{k-1}.$$

The *q-Stirling numbers of the second kind* are defined inductively for $n, k \in \mathbf{N}$ by

$$S[n,k] = \begin{cases} S[n-1, k-1] + [k]S[n-1,k] & \text{if } n, k \geq 1 \\ \delta_{n,k} & \text{if } n = 0 \text{ or } k = 0 \end{cases} \tag{2}$$

where $\delta_{n,k}$ is the Kronecker delta. These polynomials were first studied by Carlitz [1, 2] and then Gould [7].

Next, Milne [9] introduced the *dual q-Stirling numbers of the second kind* which are given by the recurrence and initial conditions

$$\hat{S}[n,k] = \begin{cases} q^{k-1}\hat{S}[n-1, k-1] + [k]\hat{S}[n-1,k] & \text{if } n, k \geq 1 \\ \delta_{n,k} & \text{if } n = 0 \text{ or } k = 0. \end{cases} \tag{3}$$

It is not hard to show that

$$\hat{S}[n,k] = q^{\binom{k}{2}} S[n,k].$$

Various authors [9, 5, 13] use S and $\tilde{S}$ to refer to the q-Stirling numbers defined by equations (3) and (2) respectively. Our reasons for reserving the simpler S notation for equation (2) are twofold: they appeared first historically and they also arise more naturally from analogs of various permutation statistics.

The connection between our versions of maj and the q-Stirling numbers is as follows.

THEOREM 1.1. *If $n, k \in \mathbf{N}$ then*:

(i) $S[n,k] = \sum_{\pi \in S(\underline{n},k)} q^{\mathrm{maj}\,\pi}$;

(ii) $\hat{S}[n,k] = \sum_{\pi \in S(\underline{n},k)} q^{\widehat{\mathrm{maj}}\,\pi}$.

PROOF. As will be our custom in the rest of this paper, we will only prove the first half of the theorem, leaving the dual version to the reader.

It is easy to see that the sum in item (i) satisfies the same boundary conditions as $S[n, k]$. To verify the recursion, consider $\pi \in S(\underline{n}, k)$ and the partition π' obtained by deleting the n from π.

If $\{n\}$ is a singleton block of $\pi = B_1/B_2/\cdots/B_k$ then, because of the standard form, we must have $B_k = \{n\}$. Thus $\pi' \in S(\underline{n-1}, k-1)$ and $\mathrm{maj}\,\pi = \mathrm{maj}\,\pi'$. Suppose, on the other hand, that n is strictly contained in some block B_i of π. Then $\pi' \in S(\underline{n-1}, k)$

and

$$\operatorname{maj} \pi = \begin{cases} \operatorname{maj} \pi' + i & \text{if } 1 \leq i < k \\ \operatorname{maj} \pi' & \text{if } i = k. \end{cases}$$

Thus

$$\sum_{\pi \in S(\underline{n},k)} q^{\operatorname{maj} \pi} = \sum_{\pi' \in S(\underline{n-1},k-1)} q^{\operatorname{maj} \pi'} + \sum_{i=0}^{k-1} \sum_{\pi' \in S(\underline{n-1},k)} q^{\operatorname{maj} \pi' + i}.$$

Since $[k]$ can be factored out of the second sum on the right, we are done. □

2. Restricted Growth Functions

Partitions can be modeled using restricted growth functions [10, 13]. The major index can then be reinterpreted in this setting.

Let $\omega = \omega_1 \omega_2 \cdots \omega_n$ be a sequence (or word) of positive integers. We say that ω is a *restricted growth* (*RG*) *function of length n* if $\omega_1 = 1$ and

$$\omega_i \leq \max_{1 \leq j < i} \omega_j + 1$$

for all i, $2 \leq i \leq n$. For example,

$$\omega = 1\,2\,1\,3\,4\,3\,3\,1\,4 \tag{4}$$

satisfies this restriction.

Let $RG(n, k)$ stand for the set of all restricted growth functions of length n such that $\max \omega = k$. It is easy to construct a bijection

$$f: S(\underline{n}, k) \to RG(n, k).$$

If $\pi \in S(\underline{n}, k)$ then $f(\pi) = \omega$, where $\omega_i = j$ whenever $i \in B_j$. The partition (1) corresponds to the RG function (4) under this map.

In what follows, we adopt the notation of White and Wachs [13]. Consider $\omega \in RG(n, k)$. Suppose that the leftmost occurrence of $j \in \underline{k}$ is in position i_j of ω and let

$$L(\omega) = \{i_1, i_2, \ldots, i_k\}.$$

Our example RG function has

$$L(\omega) = \{1, 2, 4, 5\}.$$

If ω corresponds to π via the bijection f then $L(\omega)$ is just the set of minima of the blocks of π.

Next, define two *inversion vectors*, $\mathbf{lb}(\omega)$ and $\mathbf{ls}(\omega)$, the jth components of which are given by

$$\mathbf{lb}_j(\omega) = |\{i \in L(\omega) : i < j \text{ and } \omega_i > \omega_j\}|,$$
$$\mathbf{ls}_j(\omega) = |\{i \in L(\omega) : i < j \text{ and } \omega_i < \omega_j\}|.$$

Here $\mathbf{l}, \mathbf{b}$ and $\mathbf{s}$ stand for 'left', 'bigger' and 'smaller' respectively. Continuing our running example,

$$\mathbf{lb}(\omega) = 0\,0\,1\,0\,0\,1\,1\,3\,0,$$
$$\mathbf{ls}(\omega) = 0\,1\,0\,2\,3\,2\,2\,0\,3.$$

Finally, the *major index* and *dual major index* of a word, ω, are defined by

$$\operatorname{maj} \omega = \sum_{\mathbf{lb}_j(\omega) > 0} \omega_j, \qquad \widehat{\operatorname{maj}}\, \omega = \sum_{\mathbf{ls}_j(\omega) > 0} (\omega_j - 1).$$

Interestingly, since $\omega_j - 1 = \mathbf{ls}_j(\omega)$, $\widehat{\mathrm{maj}}$ is identical to the ls inversion statistic of [13] (see Section 4). Computing these two statistics for the word in (4), we obtain

$$\mathrm{maj}\ \omega = 1 + 3 + 3 + 1 = 8,$$
$$\widehat{\mathrm{maj}}\ \omega = 1 + 2 + 3 + 2 + 2 + 3 = 13.$$

These are the same as the values obtained for the partition in (1). This is not an accident.

THEOREM 2.1. *Let $f: S(\underline{n}, k) \to RG(n, k)$ be the bijection above. Then for any $\pi \in S(\underline{n}, k)$:*
(i) $\mathrm{maj}\, f(\pi) = \mathrm{maj}\ \pi$;
(ii) $\widehat{\mathrm{maj}}\, f(\pi) = \widehat{\mathrm{maj}}\ \pi$.

PROOF. If $f(\pi) = \omega$, then $\mathbf{lb}_j(\omega) > 0$ iff $\omega_j + 1$ occurs to the left of ω_j. By definition of the function f, this corresponds to $i = \omega_j$ being in the descent set of π. Thus the constructions for maj π and maj ω coincide. □

Combining this result with Theorem 1.1 we obtain another pair of generating functions for the q-Stirling numbers. The second half of this corollary was first noted in [10] using the inversion interpretation of $\widehat{\mathrm{maj}}$.

COROLLARY 2.2. *If $n, k \in \mathbf{N}$ then:*
(i) $S[n, k] = \sum_{\omega \in RG(n,k)} q^{\mathrm{maj}\,\omega}$;
(ii) $\hat{S}[n, k] = \sum_{\omega \in RG(n,k)} q^{\widehat{\mathrm{maj}}\,\omega}$.

3. Rook Placements and Reduced Matrices

An *n-stairstep board* is a generalized chess-board consisting of n columns, where column i has length $i - 1$, $1 \leq i \leq n$, and all the columns rise from the same baseline (see Figure 1). A *rook placement*, ρ, is a way of placing non-attacking rooks on such a board, i.e. putting no two rooks in the same row or column. Let

$$SS(n, k)$$

denote the set of all placements of $n - k$ rooks on the n-stairstep board. Figure 1 gives an example of an element of $SS(9, 4)$, where a rook is indicated by an R.

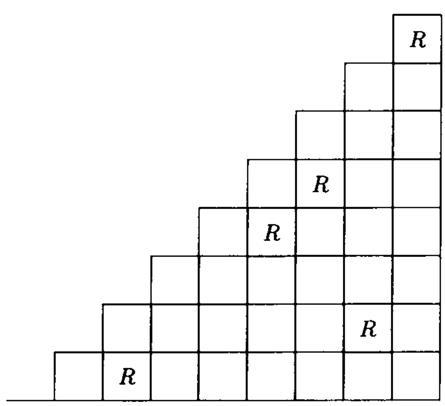

FIGURE 1. A rook placement.

It is well known that $|SS(n, k)| = S(n, k)$. We will show this with a bijection due to Wachs and White [13]. Define

$$g: S(\underline{n}, k) \to SS(n, k)$$

as follows. If $\pi \in S(\underline{n}, k)$ then construct $\rho = g(\pi)$ by placing rooks *from left to right* on the columns of the n-staircase board, i.e. starting with column 1 and ending with column n. In column j, if j is a minimum of a block of π, then leave the column empty. If $j \in B_i$ is not a minimum, then place a rook in the ith available square from the bottom of column j (a square is available if it is not attacked by a previously placed rook). The reader can check that the rook placement in Figure 1 corresponds to the partition in equation (1) under the map g. It is easy to show that this function is well defined and bijective.

We can now interpret our maj statistics in terms of rook placements. Given $\rho \in SS(n, k)$, delete all squares of the board which are strictly to the right of any rook. In Figure 2 we have indicated this process by putting a dot in each deleted square. For any rook $R \in \rho$ we let its *height* be

$$h_R = \text{number of undeleted squares below and including } R.$$

Reading from left to right, the rooks in Figure 2 have heights 1, 3, 3, 1 and 4.

The *major index* of ρ is now defined as

$$\text{maj}\,\rho = \sum_{R \in \rho} h_R,$$

where the sum is over all rooks on the board that are not at the top of their columns. Thus, in our example

$$\text{maj}\,\rho = 1 + 3 + 3 + 1 = 8$$

as usual.

Also, if C is a column of ρ then the *height* of C is

$$h_C = \begin{cases} h_R - 1 & \text{if } C \text{ contains rook } R \\ \text{number of undeleted squares in } C & \text{else} \end{cases}$$

The associated *dual major index* of ρ is

$$\widehat{\text{maj}}\,\rho = \sum_{C \in \rho} h_C.$$

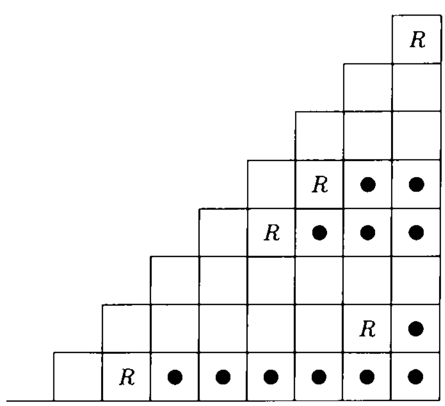

FIGURE 2. Deleting squares.

The placement of Figure 2 has

$$\widehat{\text{maj}}\,\rho = 0+1+0+2+3+2+2+0+3 = 13.$$

The next theorem should hardly come as a surprise.

THEOREM 3.1. *Let $g: S(\underline{n}, k) \to SS(n, k)$ be the bijection above. Then for any $\pi \in S(\underline{n}, k)$:*
(i) $\text{maj}\, g(\pi) = \text{maj}\, \pi$;
(ii) $\widehat{\text{maj}}\, g(\pi) = \widehat{\text{maj}}\, \pi$.

PROOF. If $g(\pi) = \rho$, then b causes a descent in π iff the rook R corresponding to b in ρ is not at the top of its column. In this case, $b \in B_i$ exactly when $h_R = i$, and the theorem follows. □

The generating function version of this result is as follows:

COROLLARY 3.2. *If $n, k \in \mathbf{N}$ then:*

(i) $S[n, k] = \sum_{\rho \in SS(n,k)} q^{\text{maj}\,\rho}$;

(ii) $\hat{S}[n, k] = \sum_{\rho \in SS(n,k)} q^{\widehat{\text{maj}}\,\rho}$.

Other statistics on the n-staircase board, the distributions of which are given by the q-Stirling numbers, will be found in [5] and [13].

Another related method of viewing partitions is via row-reduced echelon matrices, as is done in Leroux [8]. Let $RR(n, k)$ denote the set of all $k \times n$ row-reduced echelon matrices M such that:
(1) every entry of M is a 0 or a 1;
(2) there is at least one 1 in every row and exactly one 1 in every column.

It is easy to construct a bijection

$$h: S(\underline{n}, k) \to RR(n, k).$$

If $M = (m_{i,j}) = g(\pi)$ then $m_{i,j} = 1$ iff $j \in B_i$ in π. The matrix corresponding to the partition in equation (1) is

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The fact that π is in standard form corresponds to the fact that M is row-echelon.

The *major index* of the matrix m is

$$\text{maj}\, M = \sum_{m_{i,j}=1} i,$$

where the sum is restricted to those 1's which have another 1 strictly to their south-west, i.e. there exists $m_{i',j'} = 1$ with $i' > i$ and $j' < j$. The dual version is

$$\widehat{\text{maj}}\, M = \sum_{m_{i,j}=1} i - 1$$

with no restrictions on the sum.

It should be a simple matter for the reader to verify the next two results.

THEOREM 3.3. *Let $h: S(\underset{\sim}{n}, k) \to RR(n, k)$ be the bijection above. Then for any $\pi \in S(\underset{\sim}{n}, k)$:*
(i) $\operatorname{maj} h(\pi) = \operatorname{maj} \pi$;
(ii) $\widehat{\operatorname{maj}}\, h(\pi) = \widehat{\operatorname{maj}}\, \pi$.

COROLLARY 3.4. *If $n, k \in \mathbf{N}$ then:*

(i) $S[n, k] = \sum_{M \in RR(n,k)} q^{\operatorname{maj} M}$;

(ii) $\hat{S}[n, k] = \sum_{M \in RR(n,k)} q^{\widehat{\operatorname{maj}}\, M}$.

4. THE FOATA BIJECTION

After discussing inversion statistics on partitions, we introduce a bijection interchanging inv and maj. This is the partition analog of a map of Foata [3].

An *inversion* of $\pi = B_1/B_2/\cdots/B_k$ is a pair (b, B_j), where $b \in B_i$, $i<j$, and $b > \min B_j$. The *inversion number* of π, $\operatorname{inv} \pi$ is just the number of inversions in π. The partition

$$\pi = 1\,3\,8 \,/\, 2 \,/\, 4\,6\,7 \,/\, 5\,9 = B_1/B_2/B_3/B_4$$

has inversions $(3, B_2)$, $(8, B_2)$, $(8, B_3)$, $(8, B_4)$, $(6, B_4)$ and $(7, B_4)$ so $\operatorname{inv} \pi = 6$.

We could also define the *dual inversion number*, $\widehat{\operatorname{inv}}\, \pi$, to be the number of pairs (B_i, b) with $b \in B_j$, $j>i$, and $b > \min B_i$. However, it follows easily that $\widehat{\operatorname{inv}}\, \pi = \widehat{\operatorname{maj}}\, \pi$ so we have not gained anything new (cf. Section 2). Thus the results of this section and the next will apply only to statistics for $S[n, k]$.

It was noted in [10] that

$$S[n, k] = \sum_{\pi \in S(\underset{\sim}{n},k)} q^{\operatorname{inv} \pi}.$$

Thus inv and maj have the same distribution and it would be nice to have a direct combinatorial proof of this fact, i.e. a bijection $F: S(\underset{\sim}{n}, k) \to S(\underset{\sim}{n}, k)$ such that $\operatorname{inv} F(\pi) = \operatorname{maj} \pi$ for all $\pi \in S(\underset{\sim}{n}, k)$.

Define F by induction on n. By default, F is the identity when $n = 1$. If $\pi \in S(\underset{\sim}{n}, k)$ for $n > 1$, let π' be π with the n deleted. We construct $\sigma = F(\pi)$ from $\sigma' = F(\pi')$ as follows. If $\pi = B_1/B_2/\cdots/B_k$ with $B_k = \{n\}$ then let

$$\sigma = \sigma' \text{ with } \{n\} \text{ added as a singleton block.}$$

If n is strictly contained in block B_i then let

$$\sigma = \sigma' \text{ with } \{n\} \text{ added in block } \begin{cases} k-i & \text{if } 1 \leq i < k \\ k & \text{if } i = k. \end{cases}$$

In practice, to find the image of a partition π under this map, we successively compute the image of the restriction of π to the intervals $\underset{\sim}{1}, \underset{\sim}{2}, \ldots, \underset{\sim}{n}$. The process is best summarized in a table such as Table 1 for computing $F(1\,3\,8 \,/\, 2 \,/\, 4\,6\,7 \,/\, 5\,9)$.

THEOREM 4.1. *The map $F: S(\underset{\sim}{n}, k) \to S(\underset{\sim}{n}, k)$ defined above is a bijection. Furthermore, for all $\pi \in S(\underset{\sim}{n}, k)$;*
(i) $\operatorname{maj} \pi = \operatorname{inv} F(\pi)$;
(ii) $\operatorname{inv} \pi = \operatorname{maj} F(\pi)$.

TABLE 1.
The bijection F

n	π	σ
1	1	1
2	1 / 2	1 / 2
3	1 3 / 2	1 3 / 2
4	1 3 / 2 / 4	1 3 / 2 / 4
5	1 3 / 2 / 4 / 5	1 3 / 2 / 4 / 5
6	1 3 / 2 / 4 6 / 5	1 3 6 / 2 / 4 / 5
7	1 3 / 2 / 4 6 7 / 5	1 3 6 7 / 2 / 4 / 5
8	1 3 8 / 2 / 4 6 7 / 5	1 3 6 7 / 2 / 4 8 / 5
9	1 3 8 / 2 / 4 6 7 / 5 9	1 3 6 7 / 2 / 4 8 / 5 9

PROOF. It is easy to verify that F is bijective by constructing an inverse (just reverse each step in the construction of F).

To prove (i), let $\sigma = F(\pi)$ and induct on n. If π' and σ' are as above, then $\operatorname{maj} \pi' = \operatorname{inv} \sigma'$ and $\operatorname{maj} \pi = \operatorname{maj} \pi' + c$, $\operatorname{inv} \sigma' + d$, where c, d depend on the placement of n. Thus it suffices to show that $c = d$. This follows directly from a case-by-case consideration of the definition of F.

The proof of (ii) is similar. □

5. THE $r$maj STATISTIC

The $r$maj statistic of Rawlings [11] interpolates between the inversion number and the major index for permutations. The same can be done for partitions.

Let r be a positive integer. The *r-descent multiset* of $\pi \in S(\underline{n}, k)$ is

$$r\mathrm{Des}\, \pi = \{\{1^{c_1}, 2^{c_2}, \ldots, (k-1)^{c_{k-1}}\}\},$$

where c_i is the number of elements $b \in B_i$ such that $b \geq \min B_{i+1} + r$. An *r-inversion* of π is a pair (b, B_j) such that $b \in B_i$, $i < j$, and $b > \min B_j > b - r$. We let $r\mathrm{inv}\, \pi$ denote the number of r-inversions of π. Finally, the *r-major index* of π is

$$r\mathrm{maj}\, \pi = r\mathrm{inv}\, \pi + \sum_{i \in r\mathrm{Des}\, \pi} i.$$

For example, if $r = 2$ and $\pi = 1\,3\,8\,/\,2\,/\,4\,6\,7\,/\,5\,9$ then $2\,\mathrm{des}\, \pi = \{\{1, 3\}\}$ caused by the 8 and the 7; $2\mathrm{inv}\, \pi = 2$ because of the 2-inversions $(3, B_2)$ and $(6, B_4)$; thus

$$2\mathrm{maj}\, \pi = 2 + 1 + 3 = 6.$$

Clearly $r$maj reduces to maj when $r = 1$ and to inv when $r \geq n$. It turns out that $r$maj has the same distribution as these two extreme cases.

THEOREM 5.1. *If $n, k \in \mathbf{N}$ then*

$$S[n, k] = \sum_{\pi \in S(\underline{n}, k)} q^{r\mathrm{maj}\, \pi}.$$

PROOF. It is clear that the sum satisfies the same initial conditions as $S[n, k]$. For the recursion, let π' be $\pi \in S(\underline{n}, k)$ with the n deleted. If $\{n\}$ was a block, then $\pi' \in S(\underline{n-1}, k-1)$ and $r\mathrm{maj}\, \pi = r\mathrm{maj}\, \pi'$. This yields the first term of (2).

If n is strictly contained in some block B_i then $\pi' \in S(\underline{n-1}, k)$. Consider the largest index j such that $\min B_j \leq n - r$. It follows from the definition of $r$maj that

$$r\text{maj}\, \pi = \begin{cases} r\text{maj}\, \pi' + k - j + i & \text{if } 1 \leq i < j \\ r\text{maj}\, \pi' + k - i & \text{if } j \leq i \leq k. \end{cases}$$

Since $r$maj π' is increased exactly once by every integer from 0 to $k-1$, the second term of (2) is obtained. □

We can generalize the Foata bijection for partitions to one which exchanges $r$maj and $s$maj for any integers r, s as follows. We will inductively define a map $F_r: S(\underline{n}, k) \to RG(n, k)$ called the *rmaj coding*. When $n = 1$ we send $\pi = 1$ to $\omega = 1$. For $n > 1$ let π and π' be as usual with $\omega' = F_r(\pi')$. If $\{n\}$ is a singleton of π then let

$$\omega = \omega' k,$$

where juxtaposition means concatenation. If n is strictly contained in some block of π then there is a unique integer d, $0 \leq d \leq k-1$, such that $r$maj $\pi = r$maj $\pi' + d$. Let

$$\omega = \omega'(d + 1).$$

For example,

$$F_2(1\,3\,8\,/\,2\,/\,4\,6\,7\,/\,5\,9) = 1\,2\,2\,3\,4\,2\,4\,2\,1.$$

It is interesting to note that the standard encoding of partitions by RG functions (the map f of Section 2) is not one of the F_r.

The reader can easily verify the following theorem. In it, we compose functions right to left.

THEOREM 5.2. *The map $F_r: S(\underline{n}, k) \to RG(n, k)$ is a well-defined bijection. Furthermore, if $F_r(\pi) = \omega$ then*

$$r\text{maj}\, \pi = \sum_{\omega_i \notin L(\omega)} (\omega_i - 1).$$

Thus if $\sigma = F_s^{-1} F_r(\pi)$ then

$$s\text{maj}\, \sigma = r\text{maj}\, \pi.$$

6. JOINT DISTRIBUTIONS

The p, q-Stirling numbers of the second kind were first introduced by Wachs and White [13] as a generating function for the joint distribution of two inv statistics. We can now consider bivariate distributions containing maj.

Let p be another indeterminate and consider the two-variable generating function

$$S[n, k]_{p,q} = \sum_{\pi \in S(\underline{n}, k)} p^{\text{inv}\, \pi} q^{\text{maj}\, \pi}.$$

This is a new p, q-analog of $S(n, k)$ and reduces to $S[n, k]$ if either p or q is 1. Of course, we could also obtain the $S[n, k]_{p,q}$ from other pairs of statistics considered in this paper, e.g. using the ls and maj statistics on RG functions.

The p, q-analog of $k \in \mathbf{N}$ is

$$[k]_{p,q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + q^{k-1}.$$

Using the usual techniques, it is not hard to show the following:

THEOREM 6.1. *For $n, k \in \mathbf{N}$,*

$$S[n, k]_{p,q} = \begin{cases} S[n-1, k-1]_{p,q} + (1 + pq[k-1]_{p,q}) S[n-1, k]_{p,q} & \textit{if } n, k \geq 1 \\ \delta_{n,k} & \textit{else.} \end{cases}$$

It follows from the previous theorem that $S[n, k]_{p,q}$ is symmetric in p and q. In fact, we have already given a bijective proof of this fact in Theorem 4.1.

Other possible p, q-analogs include

$$\hat{S}[n, k]_{p,q} = \sum_{\pi \in S(n,k)} p^{\operatorname{inv} \pi} q^{\widehat{\operatorname{maj}} \pi}$$

and

$$\tilde{S}[n, k]_{p,q} = \sum_{\pi \in S(n,k)} p^{\operatorname{maj} \pi} q^{\widehat{\operatorname{maj}} \pi}.$$

The $\hat{S}[n, k]_{p,q}$ are the polynomials considered in [13], while the $\tilde{S}[n, k]_{p,q}$ are new. The next theorem is routine.

THEOREM 6.2. *For $n, k \in \mathbf{N}$,*

$$\hat{S}[n, k]_{p,q} = \begin{cases} q^{k-1}\hat{S}[n-1, k-1]_{p,q} + [k]_{p,q}\hat{S}[n-1, k]_{p,q} & \text{if } n, k \geq 1 \\ \delta_{n,k} & \text{else} \end{cases}$$

and

$$\tilde{S}[n, k]_{p,q} = \begin{cases} q^{k-1}\tilde{S}[n-1, k-1]_{p,q} + (p[k-1]_{pq} + q^{k-1})\tilde{S}[n-1, k]_{p,q} & \text{if } n, k \geq 1 \\ \delta_{n,k} & \text{else} \end{cases}$$

where $[k]_{pq} = 1 + pq + (pq)^2 + \cdots (pq)^{k-1}$.

7. COMMENTARY

There are other statistics the distributions of which are given by the q-Stirling numbers. The hard inversion statistics are discussed in [13]. They are so-called because it is not a simple matter to verify that they satisfy the recursions that define the q-Stirling numbers. For example, call a pair (B_i, b) a *hard inversion* of π if $b \in B_j$, $i < j$, and $\max B_i > b$. Letting $\operatorname{hinv} \pi$ be the number of hard inversions of π, we have the following non-trivial theorem:

THEOREM 7.1 ([13]). *If $n, k \in \mathbf{N}$ then*

$$S[n, k] = \sum_{\pi \in S(n,k)} q^{\operatorname{hinv} \pi}.$$

Wachs and White [personal communication] have also come up with a hard analog of the major index.

The fact that we now have a notion of descent for a partition opens up many possibilities for future research. In particular, one can define partition analogues of the Eulerian numbers. These numbers have an associated Worpitsky identity, a skew-hook formula like the one of Foulkes [4] for permutations, etc. There are also q-Stirling numbers of the first kind as introduced in [7]. These polynomials have been given statistical interpretations using permutations by Gessel [6] and using rook placements by Garsia and Remmel [5]. Both approaches only use the inversion statistic, and the rook placement version is somewhat complicated. A maj statistic also exists for these q-Stirling numbers along with a simpler interpretation using double-staircase boards. We hope to present these results on partition Eulerian numbers and the q-Stirling numbers of the first kind in the future.

Section 6 only begins to scratch the surface in terms of p, q-Stirling numbers. In particular, it would be interesting to find p, q-analogs of various identities satisfied by the ordinary Stirling numbers, an area that is currently under research by the author. For more information on these polynomials, the reader can consult [8, 12, 13].

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Bruce E. Sagan
Department of Mathematics,
Michigan State University,
East Lansing, MI 48824-1027, U.S.A.