Inversion polynomials for 321-avoiding permutations

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Abstract

We prove a generalization of a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor giving a recursion for the inversion polynomial of 321-avoiding permutations. We also answer a question they posed about finding a recursive formulas for the major index polynomial of 321-avoiding permutations. Other properties of these polynomials are investigated as well. Our tools include Dyck and 2-Motzkin paths, polyominoes, and continued fractions.

1 Introduction

The main motivation for this paper is a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor [5] about inversion polynomials for 321-avoiding permutations which we will prove in generalized form. We also answer a question they posed by giving a recursive formula for the analogous major index polynomials. We first introduce some basic definitions and notation about pattern avoidance and permutation statistics.

Call two sequences of distinct integers $\pi = a_1 \ldots a_k$ and $\sigma = b_1 \ldots b_k$ *order isomorphic* whenever $a_i < a_j$ if and only if $b_i < b_j$ for all i, j. Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. Say that $\sigma \in \mathfrak{S}_n$ *contains* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is a subsequence σ' of

σ order isomorphic to π. If σ contains no such subsequence then we say σ *avoids* π and write $\mathrm{Av}_n(\pi)$ for the set of such $\sigma \in \mathfrak{S}_n$.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and nonnegative integers, respectively. A *statistic* on $\mathfrak{S}_n$ is a function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$. One then has the corresponding generating function

$$f_n^{\mathrm{st}} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{st}\,\sigma}.$$

Two of the most ubiquitous statistics for $\sigma = b_1 \ldots b_n$ are the *inversion number*

$$\mathrm{inv}\,\sigma = \#\{(i,j) \mid i < j \text{ and } b_i > b_j\}$$

where the hash sign denotes cardinality, and the *major index*

$$\mathrm{maj}\,\sigma = \sum_{b_i > b_{i+1}} i.$$

In [16], Sagan and Savage proposed combining the study of pattern avoidance and permutations statistics by considering generating functions of the form

$$F^{\mathrm{st}}(\pi) = \sum_{\sigma \in \mathrm{Av}_n(\pi)} q^{\mathrm{st}\,\sigma} \tag{1}$$

for any pattern π and statistic st. Dokos et al. [5] were the first to carry out an extensive study of these generating functions for the inv and maj statistics. We note that when $\mathrm{st} = \mathrm{inv}$ and $\pi = 132$ we recover a q-analogue of the Catalan numbers studied by Carlitz and Riordan [2]. Work on the statistics counting fixed points and excedances has been done by Elizalde [6, 7], Elizalde and Deutsch [8], and Elizalde and Pak [9].

Our primary motivation was to prove a conjecture of Dokos et al. concerning the inversion polynomial for 321-avoiding permutations. In fact, we will prove a stronger version which also keeps track of left-right maxima. Call a_i in $\pi = a_1 \ldots a_n$ a *left-right maximum (value)* if $a_i = \max\{a_1, \ldots, a_i\}$. We let

$$\mathrm{Lrm}\,\pi = \{a_i \mid a_i \text{ is a left-right maximum}\}$$

and $\mathrm{lrm}\,\pi = \#\,\mathrm{Lrm}\,\pi$. Consider the generating function

$$I_n(q,t) = \sum_{\sigma \in \mathrm{Av}_n(321)} q^{\mathrm{inv}\,\sigma} t^{\mathrm{lrm}\,\sigma}. \tag{2}$$

Note that since $\#\,\mathrm{Av}_n(321) = C_n$, the nth Catalan number, this polynomial is a q,t-analogue of C_n. Our main result is a recursion for $I_n(q,t)$. The case $t = 1$ was a conjecture of Dokos et al.

Theorem 1.1. *For $n \geq 1$,*

$$I_n(q,t) = tI_{n-1}(q,t) + \sum_{k=0}^{n-2} q^{k+1} I_k(q,t) I_{n-k-1}(q,t).$$

The rest of this paper is structured as follows. In the next section we will give a direct bijective proof of Theorem 1.1 using 2-Motzkin paths. The following two sections will explore related ideas involving Dyck paths, including a combinatorial proof of a formula of Fürlinger and Hofbauer [11] and two new statistics which are closely related to inv. Sections 5 and 6 are devoted to polyominoes. First, we give a second proof of Theorem 1.1 using work of Cheng, Eu, and Fu [3]. Next we derive recursions for a major index analogue, $M_n(q,t)$, of (2), thus answering a question posed by Dokos et al. in their paper. In Section 7, we explore further properties of $M_n(q,t)$, including symmetry, unimodality, and its modulo 2 behavior. The final two sections are concerned with continued fractions. We begin by giving a third proof of Theorem 1.1 using continued fractions and comparing it with a result of Krattenthaler [14]. In fact, this demonstration is even more general since it also keeps track of the number of fixed points. And in Section 9 we reprove and then generalize a theorem of Simion and Schmidt [17] concerning the signed-enumeration of 321-avoiding permutations.

2 A proof of Theorem 1.1 using 2-Motzkin paths

Our first proof of Theorem 1.1 will use 2-Motzkin paths. Let U (up), D (down), and L (level) denote vectors in $\mathbb{Z}^2$ with coordinates $(1,1)$, $(1,-1)$, and $(1,0)$, respectively. A *Motzkin path of length* n, $M = s_1 \ldots s_n$, is a lattice path where each step s_i is U, D, or L and which begins at the origin, ends on the x-axis, and never goes below $y = 0$. A 2-*Motzkin path* is a Motzkin path where each level step has been colored in one of two colors which we will denote by L_0 and L_1. We will let $\mathcal{M}_n^{(2)}$ denote the set of 2-Motzkin paths of length n.

It will be useful to have two vectors to keep track of the values and positions of left-right maxima. If $\sigma = b_1 \ldots b_n \in \mathfrak{S}_n$ then let

$$\operatorname{val}\sigma = (v_1, \ldots, v_n)$$

where

$$v_i = \begin{cases} 1 & \text{if } i \text{ is a left-right maximum of } \sigma, \\ 0 & \text{else.} \end{cases}$$

Also define

$$\operatorname{pos}\sigma = (p_1, \ldots, p_n)$$

where

$$p_i = \begin{cases} 1 & \text{if } b_i \text{ is a left-right maximum of } \sigma, \\ 0 & \text{else.} \end{cases}$$

By way of example, if $\sigma = 361782495$ then we have $\operatorname{val}\sigma = (0,0,1,0,0,1,1,1,1)$ and $\operatorname{pos}\sigma = (1,1,0,1,1,0,0,1,0)$. Note that for any permutation $v_n = p_1 = 1$.

We will need the following lemma which collects together some results from the folklore of pattern avoidance. Since they are easy to prove, the demonstration will be omitted.

Lemma 2.1. *Suppose* $\sigma \in \mathfrak{S}_n$.

(a) *We have* $\sigma \in \operatorname{Av}_n(321)$ *if and only if the elements of* $[n] - \operatorname{Lrm}\sigma$ *form an increasing subsequence of* σ.

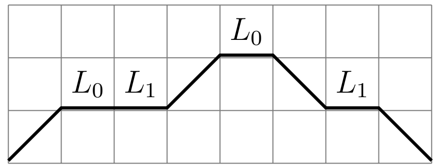


Figure 1: The Motzkin path associated with $\sigma = 361782495$

(b) *Suppose we are given* 0-1 *vectors* $v = (v_1, \ldots, v_n)$ *and* $p = (p_1, \ldots, p_n)$ *with the same positive number of ones. Then* $v = \operatorname{val} \sigma$ *and* $p = \operatorname{pos} \sigma$ *for some* $\sigma \in \mathfrak{S}_n$ *if and only if, for every index* i, $1 \le i \le n$, *the number of ones in* $p_1, \ldots, p_i$ *is greater than the number in* $v_1, \ldots, v_{i-1}$. *In this case, because of part (a), there is a unique such* $\sigma \in \operatorname{Av}_n(321)$. □

Note that in (b) the cases where $i = 1$ and $i = n$ force $p_1 = 1$ and $v_n = 1$, respectively.

First proof of Theorem 1.1. We will construct a bijection $\mu : \operatorname{Av}_n(321) \to \mathcal{M}_{n-1}^{(2)}$ as follows. Given $\sigma \in \operatorname{Av}_n(321)$ with $\operatorname{val} \sigma = (v_1, \ldots, v_n)$ and $\operatorname{pos} \sigma = (p_1, \ldots, p_n)$, we let $\mu(\sigma) = M = s_1 \ldots s_{n-1}$ where

$$s_i = \begin{cases} U & \text{if } v_i = 0 \text{ and } p_{i+1} = 1, \\ D & \text{if } v_i = 1 \text{ and } p_{i+1} = 0, \\ L_0 & \text{if } v_i = p_{i+1} = 0, \\ L_1 & \text{if } v_i = p_{i+1} = 1. \end{cases}$$

Continuing the example from the beginning of the section, $\sigma = 361782495$ would be mapped to the path in Figure 1.

We must first show that μ is well defined in that if $M = \mu(\sigma)$ then M ends on the x-axis and stays weakly above it the rest of the time. In other words, we want the number of U's in any prefix of M to be at least as great as the number of D's, with equality at the finish. This now follows from the definition of the s_i and the first two sentences of Lemma 2.1 (b).

We must also check that μ is a bijection. The fact that it is injective is an immediate consequence of the definition of the s_i and the third sentence of Lemma 2.1 (b). Since $\# \operatorname{Av}_n(321) = C_n = \# \mathcal{M}_{n-1}^{(2)}$, we also have bijectivity.

If $\mu(\sigma) = M$ then we claim that

$$\begin{aligned} \operatorname{lrm} \sigma &= \#U(M) + \#L_1(M) + 1, & (3) \\ \operatorname{inv} \sigma &= \#D(M) + \#L_0(M) + \operatorname{area} M, & (4) \end{aligned}$$

where $\operatorname{area} M$ is the area between M and the x-axis, $U(M)$ is the set of steps equal to U in M, and similarly for the other types of steps. The first equation follows from the definition of M and the fact that $\operatorname{lrm} \sigma$ is the number of ones in $\operatorname{pos} \sigma$. The $+1$ is because M has length $n-1$ and we always have $p_1 = 1$.

To prove the equation for inv, we will induct on n. Note that every $M \in \mathcal{M}_{n-1}^{(2)}$, where $n \ge 2$, can be uniquely decomposed in one of the following ways:

(i) $M = L_0 N$, where $N \in \mathcal{M}_{n-2}^{(2)}$,

(ii) $M = L_1 N$, where $N \in \mathcal{M}_{n-2}^{(2)}$.

(iii) $M = UNDO$, where $N \in \mathcal{M}^{(2)}_{k-1}$ and $O \in \mathcal{M}^{(2)}_{n-k-2}$ for some $1 \le k \le n-2$.

Suppose (4) holds for N in case (i), and suppose $\mu(\pi) = N$. If we have $\operatorname{val}\pi = (v_1, \ldots, v_{n-1})$ and $\operatorname{pos}\pi = (p_1, \ldots, p_{n-1})$ then adding L_0 forces the vectors to change to $\operatorname{val}\sigma = (0, v_1, \ldots, v_{n-1})$ and $\operatorname{pos}\sigma = (1, 0, p_2, \ldots, p_{n-1})$. It follows that if $\pi = a_1 \ldots a_{n-1}$ then $\sigma = (a_1{+}1)1(a_2{+}1)\ldots(a_{n-1}{+}1)$. So both the left and right sides of (4) go up by one when passing from π to σ and equality is preserved. Similar arguments shows that both sides stay the same in case (ii), and both go up by $k+1$ in case (iii). So equality is maintained in all cases.

From what we have shown, it suffices to show that

$$I'_n(q,t) = \sum_{P \in \mathcal{M}^{(2)}_{n-1}} q^{\#D(P)+\#L_0(P)+\operatorname{area} P} t^{\#U(P)+\#L_1(P)+1}$$

satisfies the recurrence in Theorem 1.1. Considering the three cases above, in (i) we get a contribution of $qI'_{n-1}(q,t)$ to $I'_n(q,t)$ which corresponds to the $k=0$ term of the sum. Similarly, case (ii) contributes $tI'_{n-1}(q,t)$. Finally, in (iii) the piece UND contributes $tq^{k+1}I_k(q,t)$ since when lifting N the area is increased by k, and both $\#U(N)$ and $\#D(N)$ are increased by one. Also O contributes $I_{n-k-1}(q,t)/t$ since there is a $+1$ in the exponent of t for both $I_k(q,t)$ and $I_{n-k-1}(q,t)$, but we only want one such. Combining these contributions proves the recursion. □

3 Dyck paths and an equation of Fürlinger and Hofbauer

In this section we will prove a formula of Fürlinger and Hofbauer [11, equation (5.5)] which is closely related to Theorem 1.1. In fact, we will show in Section 5 that this equation can be used to prove our main theorem. Our proof of the Fürlinger-Hofbauer result will be combinatorial using Dyck paths, whereas the one given in [11] is by algebraic manipulation of generating functions. Our proof has the interesting feature that it uses a nonstandard decomposition of Dyck paths which will also be useful in the next section.

Let $P = s_1 \ldots s_{2n}$ be a Dyck path of semilength n and let $\mathcal{D}_n$ be the set of all such P. We will freely go back and forth between three standard interpretations of such paths. In the first, P consists of n U-steps and n D-steps starting at the origin and staying weakly above the x-axis. It the second, there are n north steps, $N = (0,1)$, and n east steps, $E = (1,0)$, beginning at the origin and staying weakly above the line $y = x$. In the last, we have n zeros and n ones with the number of zeros in any prefix of P being at least as great as the number of ones. In this last interpretation, we can apply all the usual permutation statistics defined in the same way as they were when there were no repetitions. In particular, we will need the *descent set* of P

$$\operatorname{Des} P = \{i \mid a_i > a_{i+1}\}$$

and the *descent number* $\operatorname{des} P = \#\operatorname{Des} P$. A descent of P as a bit string corresponds to a *valley* of P in the first interpretation, i.e., a factor of the form DU. We will also need the dual notion of a *peak*, which is a factor UD.

We also need to define one of the analogues of the Catalan numbers studied by Fürlinger and Hofbauer. Given a Dyck path $P = s_1 \ldots s_{2n}$ we let $|P|_0$ and $|P|_1$ be the number of zeros and number of ones in P, respectively. More generally we will write $|w|_A$ for the number of occurrences of A in the word w for any A and w. Now let

$$\mathfrak{p}_i(w) = s_1 \dots s_i \tag{5}$$

be w's prefix of length i. Define

$$\begin{aligned}\alpha(P) &= \sum_{i \in \operatorname{Des} P} |\mathfrak{p}_i(P)|_0, \\ \beta(P) &= \sum_{i \in \operatorname{Des} P} |\mathfrak{p}_i(P)|_1.\end{aligned}$$

Note that $\alpha(P) + \beta(P) = \operatorname{maj} P$. Now consider the generating function

$$C_n(t) = C_n(a, b; t) = \sum_{P \in \mathcal{D}_n} a^{\alpha(P)} b^{\beta(P)} t^{\operatorname{des} P}. \tag{6}$$

Theorem 3.1 (Fürlinger and Hofbauer [11])**.** *We have*

$$C_n(t) = C_{n-1}(abt) + bt \sum_{k=0}^{n-2} a^{k+1} C_k(abt) C_{n-k-1}((ab)^{k+1}t). \tag{7}$$

Proof. We will first define a bijection $\delta : \biguplus_{k=0}^{n-1} \mathcal{D}_k \times \mathcal{D}_{n-k-1} \to \mathcal{D}_n$. Given two Dyck paths

$$Q = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \dots U^{a_s} D^{b_s} \in \mathcal{D}_k \quad \text{and} \quad R = U^{c_1} D^{d_1} U^{c_2} D^{d_2} \dots U^{c_t} D^{d_t} \in \mathcal{D}_{n-k-1}$$

where all exponents are positive, we will combine them to create a Dyck path $P = \delta(Q, R) \in \mathcal{D}_n$ as follows. (When Q is empty, the same definition works with the convention that $a_1 = b_1 = 0$). There are two cases:

1. If $R = \emptyset$, then
$$P = U^{a_1+1} D^{b_1+1} U^{a_2} D^{b_2} \dots U^{a_s} D^{b_s}.$$

2. If $R \neq \emptyset$, then
$$P = U^{a_1+1} D U^{a_2} D^{b_1} U^{a_3} D^{b_2} \dots U^{a_s} D^{b_{s-1}} U^{c_1} D^{b_s + d_1} U^{c_2} D^{d_2} \dots U^{c_t} D^{d_t}.$$

For example,
$$\delta(U^3 D^2 U D^2, U D U^2 D U D^2) = U^4 D U D^2 U D^3 U^2 D U D^2,$$
as illustrated in Figure 2. The path P is given by the solid lines while Q is dashed and R is dotted, with Q being shifted to start at $(2, 0)$ and R concatenated directly after, starting at $(10, 0)$. Note that this puts the peaks of Q at exactly the same position as the valleys of the first part of P, and makes R and P coincide after the first peak of R

To show that δ is bijective, we construct its inverse. Suppose that

$$P = U^{i_1} D^{j_1} U^{i_2} D^{j_2} \dots U^{i_k} D^{j_k}.$$

Again, there are two cases for computing $\delta^{-1}(P) = (Q, R)$.

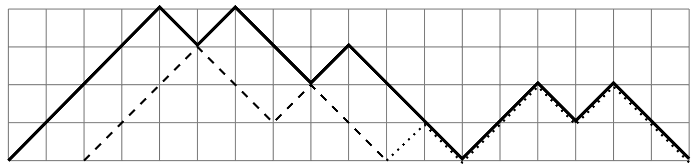


Figure 2: The decomposition of Dyck paths

1. If $j_1 \geq 2$, then
$$Q = U^{i_1-1}D^{j_1-1}U^{i_2}D^{j_2}\dots U^{i_k}D^{j_k} \quad \text{and} \quad R = \emptyset.$$

2. If $j_1 = 1$, let s be the smallest index such that $i_1 + i_2 + \cdots + i_s \leq j_2 + j_3 + \cdots + j_{s+1}$ (note that $s < k$), and let $\epsilon = j_2 + j_3 + \cdots + j_{s+1} - (i_1 + i_2 + \cdots + i_s)$. Then
$$Q = U^{i_1-1}D^{j_2}U^{i_2}D^{j_3}\dots U^{i_s}D^{j_{s+1}-\epsilon-1} \quad \text{and} \quad R = U^{i_{s+1}}D^{\epsilon+1}U^{i_{s+2}}D^{j_{s+2}}\dots U^{i_k}D^{i_k}.$$

The first case is easy to understand as you just shorten the first peak of P which had been lengthened by δ. For the second case, the reader may find it useful to consult Figure 2 again. As in defining δ, Q is the Dyck path that starts at $(2,0)$ and has the peaks at the valleys of P. At some point this will no longer be possible because Q would have to go under the x-axis. Q ends at the point on the x-axis just before it would be forced to go negative, and R starts at that point. The path R begins with up-steps until it hits P, and then coincides with P for the rest of the path.

We now use our bijection to prove the theorem. If $P \in \mathcal{D}_n$ and $\delta^{-1}(P) = (Q,R)$, then the term $C_{n-1}(abt)$ corresponds to the case when $R = \emptyset$, since then adding the extra UD to the first peak moves all the valleys over by one unit. When $R \neq \emptyset$, suppose $Q \in \mathcal{D}_k$ and $R \in \mathcal{D}_{n-k-1}$. The factor $C_{n-k-1}((ab)^{k+1}t)$ comes from the fact that all the valleys of R become valleys of P, each one having $k+1$ additional steps U and D to their left. The term $C_k(abt)$ accounts for the fact that when the valleys of Q become valleys of P, they have an additional U and D inserted to their left, namely the first D in P and the U preceding it, and the term abt accounts for the extra valley started by this D. Finally, the factor a^k comes from the fact that when the k up-steps of Q are put in P, they are shifted one valley to the left of the corresponding valley in P. Thus each of these steps moves the corresponding valley one position to the right. $\square$

The decomposition in Theorem 3.1 is based on the structure of the first peak of P. If we consider the last peak of P, we will get the following equation which does not appear in [11].

Theorem 3.2. *We have*
$$C_n(t) = C_{n-1}(t) + a^{n-1}t\sum_{k=1}^{n-1} b^k C_k(t) C_{n-k-1}((ab)^k t).$$

Proof. We define a second bijection $\Delta : \biguplus_{k=0}^{n-1} \mathcal{D}_k \times \mathcal{D}_{n-k-1} \to \mathcal{D}_n$ as follows. Given two Dyck paths
$$R = U^{c_1}D^{d_1}U^{c_2}D^{d_2}\dots U^{c_t}D^{d_t} \in \mathcal{D}_k \quad \text{and} \quad Q = U^{a_1}D^{b_1}U^{a_2}D^{b_2}\dots U^{a_s}D^{b_s} \in \mathcal{D}_{n-k-1}$$

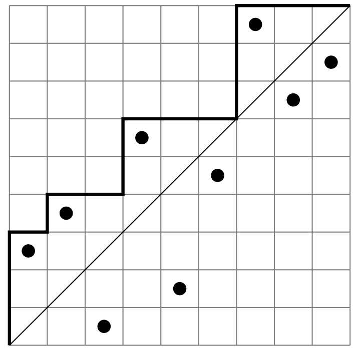


Figure 3: The bijection Γ

1. If $R = \emptyset$, then
$$P = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \dots U^{a_s+1} D^{b_s+1}.$$

2. If $R \neq \emptyset$, then
$$P = U^{c_1} D^{d_1} U^{c_2} D^{d_2} \dots U^{c_t+a_1} D^{d_t} U^{a_2} D^{b_1} U^{a_3} D^{b_2} \dots U^{a_s} D^{b_{s-1}} U D^{b_s+1}.$$

Now using similar arguments to those in the proof of Theorem 3.1, we get the desired result. $\square$

4 The sumpeaks and sumtunnels statistics

In this section we discuss another pair of (new) statistics on Dyck paths, which are closely related to the inv statistic on 321-avoiding permutations. In fact, these two statistics are equidistributed over $\mathfrak{S}_n$, as we will show. First we will need some definitions and notation.

Let $\operatorname{npea} P$ denote the number of peaks of P, and note that $\operatorname{npea} P = \operatorname{des} P + 1$ for any non-empty Dyck path P. Also define the *height* of a peak $p = UD = s_i s_{i+1}$ in a Dyck path $P = s_1 \dots s_{2n}$ to be

$$\operatorname{ht}(p) = |\mathfrak{p}_i(P)|_U - |\mathfrak{p}_i(P)|_D, \tag{8}$$

where $\mathfrak{p}_i$ is as in equation (5). In the interpretation of Dyck paths using steps U and D, $\operatorname{ht}(p)$ is the y-coordinate of the highest point of p. In Figure 2, the peaks of P have heights $4, 4, 3, 2$, and 2.

To make a connection with permutations $\pi = a_1 \dots a_n$, we consider the *diagram* of π, which is a square grid with a dot in column j at height a_j for all $j \in [n]$. Figure 2 displays the permutation $\pi = 341625978 \in \operatorname{Av}_9(321)$. We will now describe a bijection $\Gamma : \operatorname{Av}_n(321) \to \mathcal{D}_n$ which appeared in [6] (where it is denoted by ψ_3), and in a slightly different form in [14]. Imagine a light shining from the northwest of the diagram of π so that each dot casts a shadow with sides parallel to the axes. Consider the lattice path P formed by the boundary of the union of these shadows. (This is the same procedure as used by Viennot [19] in his geometric version of the Robinson-Schensted correspondence.) Define $\Gamma(\pi) = P$. Again, Figure 2 illustrates the process. Using Lemma 2.1,

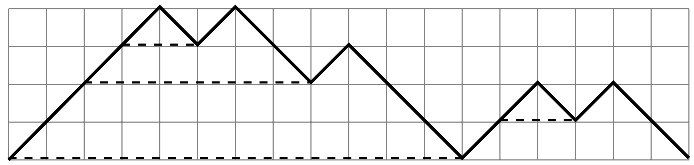


Figure 4: The tunnels of a Dyck path

one can prove that for any permutation $\pi \in \mathfrak{S}_n$, the path P will stay above $y = x$ and so be a Dyck path. Lemma 2.1 also shows that if one restricts to $\pi \in \mathrm{Av}_n(321)$, then this map becomes a bijection. The inv and lrm statistics on π translate nicely under Γ.

Proposition 4.1. *If $\Gamma(\pi) = P$, then*

(a) $\operatorname{lrm} \pi = \operatorname{npea} P$,

(b) $\operatorname{inv} \pi = \sum_p (\operatorname{ht}(p) - 1)$ *where the sum is over all peaks p of P.*

Proof. For (a), just note that every left-right maximum of π is associated with a peak $p = NE$ of P consisting of two of the edges of the square containing the maximum. And this correspondence is clearly reversible.

For (b), since $\pi = a_1 \dots a_n$ avoids 321, each inversion (i, j) of π has the property that a_i is a left-right maximum but a_j is not. There is a two-to-one correspondence between elements of π and steps of P, where a_i corresponds to the pair (N_i, E_i) which are the projections horizontally and vertically onto P, respectively. Now (i, j) is an inversion if and only if N_j comes before N_i and E_j comes after E_i. Thus it follows from equation (8) (with U and D replaced by N and E, respectively) that the element a_i corresponding to a peak p causes inversions with exactly $\operatorname{ht}(p) - 1$ elements a_j. The -1 comes from the fact that (i, i) is not an inversion. Summing over all peaks completes the proof. □

We will show at the end of Section 6 that the bijection Γ can be used to give alternative proofs of Theorems 1.1 and 6.2.

Because of its appearance in the previous proposition, we define a new statistic *sumpeaks* on Dyck paths P by

$$\operatorname{spea} P = \sum_p (\operatorname{ht}(p) - 1)$$

where the sum is over all peaks p of P. There is another statistic that we will now define which is equidistributed with sumpeaks.

If P is a Dyck path with steps U, D and $v = DU = s_j s_{j+1}$ is a valley of P then its *height*, $\operatorname{ht}(v)$, is the y-coordinate of its lowest point. For each valley, v, there is a corresponding *tunnel*, which is the factor $T = s_i \dots s_j$ of P where s_i is the step after the first intersection of P with the line $y = \operatorname{ht}(v)$ to the left of s_j. The tunnels for the Dyck path in Figure 4 are indicated with dashed lines. In every tunnel, $j - i$ is an even number, so we define the *sumtunnels* statistic to be

$$\operatorname{stun} P = \sum_{T = s_i \dots s_j} (j - i)/2$$

where the sum is over all tunnels T of P. In Figure 4, we have $\operatorname{stun} P = (12 + 6 + 2 + 2)/2 = 11$. It turns out that the sumpeaks and sumtunnels statistics are equidistributed over $\mathcal{D}_n$.

Theorem 4.2. *For any $n \geq 1$,*

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{spea} P} t^{\operatorname{npea} P} = \sum_{P \in \mathcal{D}_n} q^{\operatorname{stun} P} t^{n - \operatorname{des} P}.$$

Proof. For $P \in \mathcal{D}_n$, let $\overline{\operatorname{des}}\, P = n - \operatorname{des} P$ for convenience. Recall that $\operatorname{des} P$ is the number of valleys of P. It suffices to define a bijection $h : \mathcal{D}_n \to \mathcal{D}_n$ such that for any $P \in \mathcal{D}_n$ we have

$$\operatorname{spea} P = \operatorname{stun} h(P) \quad \text{and} \quad \operatorname{npea} P = \overline{\operatorname{des}}\, h(P). \tag{9}$$

Let $\delta^{-1}(P) = (Q, R)$ where δ is the bijection of the Section 3. We inductively define h by $h(\emptyset) = \emptyset$ and for $n \geq 1$

$$h(P) = \begin{cases} UDh(Q) & \text{if } R = \emptyset, \\ Uh(R)D & \text{if } Q = \emptyset, \\ Uh(Q)Dh(R) & \text{else.} \end{cases}$$

To see that h has an inverse, it suffices to check that given $P' \in \mathcal{D}_n$ we can tell which of the three cases above P' must fall into for $|P'| \geq 4$. (When $|P'| = 2$ then bijectivity is clear since there is only one Dyck path of this length.) The first case contains all P' starting with a single U. The second case covers all P' that are strictly above the x-axis between the first and last lattice points which forces them to start with at least two U's. And the last case contains those paths which start with at least two U's and intersect the x-axis before the final vertex.

We now verify equation (9) by induction on n. It is easy to verify for $n = 1$. For greater n, let

$$P = U^{i_1} D^{j_1} U^{i_2} D^{j_2} \ldots U^{i_k} D^{j_k},$$

where $i_1, j_1, \ldots, i_k, j_k$ are positive. We have three cases.

If $j_1 \geq 2$, then $Q = U^{i_1 - 1} D^{j_1 - 1} U^{i_2} D^{j_2} \ldots U^{i_k} D^{j_k}$ and $R = \emptyset$. So, comparing P and Q and using the induction hypothesis,

$$\begin{aligned} \operatorname{spea} P &= \operatorname{spea} Q + 1 = \operatorname{stun} h(Q) + 1 = \operatorname{stun} UDh(Q) = \operatorname{stun} h(P), \\ \operatorname{npea} P &= \operatorname{npea} Q = \overline{\operatorname{des}}\, h(Q) = \overline{\operatorname{des}}\, UDh(Q) = \overline{\operatorname{des}}\, h(P). \end{aligned}$$

If $j_1 = i_1 = 1$, then $Q = \emptyset$ and $R = U^{i_2} D^{j_2} \ldots U^{i_k} D^{j_k}$. Using similar reasoning to the first case,

$$\begin{aligned} \operatorname{spea} P &= \operatorname{spea} R = \operatorname{stun} h(R) = \operatorname{stun} Uh(R)D = \operatorname{stun} h(P), \\ \operatorname{npea} P &= \operatorname{npea} R + 1 = \overline{\operatorname{des}}\, h(R) + 1 = \overline{\operatorname{des}}\, Uh(R)D = \overline{\operatorname{des}}\, h(P). \end{aligned}$$

If $j_1 = 1$ and $i_1 \geq 2$ then, keeping the notation in the definition of δ,

$$Q = U^{i_1 - 1} D^{j_2} U^{i_2} D^{j_3} \ldots U^{i_s} D^{j_{s+1} - \epsilon - 1} \quad \text{and} \quad R = U^{i_{s+1}} D^{\epsilon + 1} U^{i_{s+2}} D^{j_{s+2}} \ldots U^{i_k} D^{i_k}.$$

The last peaks of P coincide with all but the first peak of R. The first peak of R and the peaks of Q are in bijection with the rest of the peaks of P where a peak of P corresponds to the peak of Q (or the first peak of R) which is closest on its right. Let $p_1, \ldots, p_{s+1}$ be these peaks of P, corresponding to peaks $q_1, \ldots, q_s$ in Q and $r_1 \overset{\text{def}}{=} q_{s+1}$ in R. Then

$$\operatorname{ht}(p_k) = \begin{cases} \operatorname{ht}(q_1) + 1 & \text{if } k = 1, \\ \operatorname{ht}(q_k) + j_k & \text{if } 1 < k \leq s, \\ \operatorname{ht}(q_{s+1}) + j_{s+1} - \epsilon - 1 & \text{if } k = s + 1. \end{cases}$$

Thus we have

$$\sum_{k=1}^{s+1} \mathrm{ht}(p_k) = \sum_{k=1}^{s+1} \mathrm{ht}(q_k) + j_2 + \cdots + j_{s+1} - \epsilon = \sum_{k=1}^{s+1} \mathrm{ht}(q_k) + i_1 + \cdots + i_s.$$

Hence

$$\begin{aligned} \operatorname{spea} P &= \operatorname{spea} Q + \operatorname{spea} R + \sum_{k=1}^{s} i_k \\ &= \operatorname{stun} h(Q) + \operatorname{stun} h(R) + \sum_{k=1}^{s} i_k \\ &= \operatorname{stun} Uh(Q)Dh(R) \\ &= \operatorname{stun} h(P), \end{aligned}$$

where the third equality comes from the fact that $Uh(Q)Dh(R)$ has exactly one more tunnel than the union of the tunnels of Q and R, namely the tunnel from the new U to the new D, and that tunnel has semilength $\sum_{k=1}^{s} i_k$. Additionally,

$$\operatorname{npea} P = \operatorname{npea} Q + \operatorname{npea} R = \overline{\operatorname{des}}\, h(Q) + \overline{\operatorname{des}}\, h(R) = \overline{\operatorname{des}}\, Uh(Q)Dh(R) = \overline{\operatorname{des}}\, h(P),$$

completing the proof. □

5 A proof of Theorem 1.1 using polyominoes

In this section we will give a second proof of our main theorem using Theorem 3.1, another result of Fürlinger and Hofbauer, and polyominoes. In particular, we will need a bijection Υ first defined by Cheng, Eu, and Fu [3] between shortened polyominoes and 321-avoiding permutations. We first need to define some terms.

A *parallelogram polyomino* is a pair (U, V) of lattice paths using steps N and E such that

- U and V begin at the same vertex and end at the same vertex, and
- U stays strictly above V except at the beginning and end vertices.

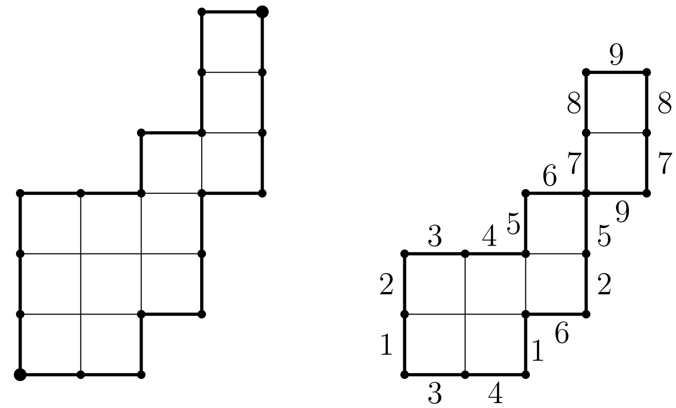
In Figure 5(a) we have $U = NNNEEENENNE$ and $V = EENENNENNN$. Let $\mathcal{P}_n$ denote the set of all parallelogram polyominoes with $|U| = |V| = n$. Note that if $U = s_1 \ldots s_n$ and $V = t_1 \ldots t_n$ then $s_1 = t_n = N$, and $s_n = t_1 = E$. Define two statistics

$$\begin{aligned} \operatorname{area}(U,V) &= \text{the area contained inside } (U,V), \\ \operatorname{col}(U,V) &= \text{the number of columns spanned by } (U,V). \end{aligned}$$

Returning to our example, $\operatorname{area}(U, V) = 12$ and $\operatorname{col}(U, V) = 4$. Consider the generating function

$$P_n(q,t) = \sum_{(U,V)\in\mathcal{P}_n} q^{\operatorname{area}(U,V)} t^{\operatorname{col}(U,V)}.$$

Another result of Fürlinger and Hofbauer, which we state here without proof, shows that this polynomial is closely related to $C_n(a, b; t)$ as defined in equation (7).



(a) A polyomino in $\mathcal{P}_{10}$ (b) A polyomino in $\mathcal{H}_9$

Figure 5: Parallelogram and shortened polyominoes

Theorem 5.1 (Fürlinger and Hofbauer [11]). *We have*

$$P_{n+1}(q,t) = q^n t C_n(q, q^{-1}; t)$$

for all $n \geq 0$. □

We will also need another type of polyomino. Define a *shortened polyomino* to be a pair (P, Q) of N, E lattice paths satisfying

- P and Q begin at the same vertex and end at the same vertex, and
- P stays weakly above Q and the two paths can share E-steps but not N-steps.

Figure 5(b) shows such a polyomino. We denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$.

We can now define the map $\Upsilon : \mathcal{H}_n \to \mathrm{Av}_n(321)$. Given $(P, Q) \in \mathcal{H}_n$, label the steps of P with the numbers $1, \ldots, n$ from south-west to north-east. Each step of P is paired with the projection of that step onto Q. Give each step of Q the same label as its pair. Then reading the labels on Q from south-east to north-west gives a permutation $\sigma = \Upsilon(P, Q)$. In Figure 5(b), $\sigma = 341625978$. The next result compares our statistics on $\mathcal{H}_n$ and $\mathrm{Av}_n(321)$

Theorem 5.2 (Cheng-Eu-Fu [3]). *The map* $\Upsilon : \mathcal{H}_n \to \mathrm{Av}_n(321)$ *is a well-defined bijection such that if* $\Upsilon(P, Q) = \sigma$ *then*

(a) $\mathrm{area}(P, Q) = \mathrm{inv}\,\sigma$, *and*

(b) $\mathrm{col}(P, Q) = \mathrm{lrm}\,\sigma$.

Proof. The fact that Υ is a well-defined bijection and part (a) were proved in [3], so we will only sketch the main ideas here. If $\Upsilon(P, Q) = \sigma$, then the left-right maxima of σ will label the E steps of Q. The positions of these maxima in σ are the same as their positions on Q. Thus, as we saw in Lemma 2.1(b), this data will determine a unique 321-avoiding permutation provided that the prefix condition is satisfied. And that condition is ensured by the second item in the definition of a shortened polyomino. Thus we have a bijection.

Now suppose $\sigma = a_1 \ldots a_n$ and that we have an inversion $a_i > a_j$ where $i < j$. In that case a_i and a_j will label an E-step and an N-step of Q, respectively, with the N-step coming later on the path. One can then show that there will be a square inside (P, Q) due north of a_i and due west of a_j corresponding to the inversion. This process is reversible, so there is a bijection between inversions of σ and squares inside (P, Q), proving part (a) of the theorem. And part (b) follows from the already-noticed fact that the left-right maxima of σ are in bijection with the E-steps of Q. $\square$

The final ingredient is a simple bijection between $\mathcal{P}_{n+1}$ and $\mathcal{H}_n$: If $(U, V) \in \mathcal{P}_{n+1}$ then contracting the first step of U and the last step of V (both of which are N-steps) gives $(P, Q) \in \mathcal{H}_n$. The polyomino in Figure 5(b) is gotten by shortening the one in 5(a) in this manner. If shortening (U, V) gives (P, Q) then we clearly have

$$\begin{aligned} \operatorname{area}(U, V) &= \operatorname{area}(P, Q) + \operatorname{col}(P, Q), && (10) \\ \operatorname{col}(U, V) &= \operatorname{col}(P, Q). && (11) \end{aligned}$$

Second proof of Theorem 1.1. Theorem 5.2 together with equations (10) and (11) give $I_n(q, t) = P_{n+1}(q, t/q)$. Combining this with Theorem 5.1 yields

$$I_n(q, t) = \begin{cases} q^{n-1} t C_n(q, 1/q; t/q) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Now in Theorem 3.1 we replace a, b, and t by $q, 1/q$, and t/q , respectively. Multiplying both sides by $q^{n-1}t$ and rewriting everything in terms of the corresponding inversion polynomials finishes the proof. $\square$

6 A major index polynomial recursion

In the paper of Dokos et al., they asked for a recursion for the 321-avoiding major index polynomial which is defined by equation (1) with st = maj and $\pi = 321$. The purpose of this section is to give such a recurrence relation using polyominoes.

Consider the polynomial

$$M_n(q, t) = \sum_{\sigma \in \operatorname{Av}_n(321)} q^{\operatorname{maj} \sigma} t^{\operatorname{des} \sigma}. \tag{12}$$

Using the description of the bijection $\Upsilon : \mathcal{H}_n \to \operatorname{Av}_n(321)$ given in the proof of Theorem 5.2, it is clear that each descent of σ corresponds to a factor EN of Q where $\Upsilon^{-1}(\sigma) = (P, Q)$ and vice-versa. Since the position of the descent in σ is the same as the position of the factor on Q, we have $\operatorname{Des} \sigma = \operatorname{Des} Q$ where, as usual, Q is identified with the bit string obtained by replacing N and E by 0 and 1, respectively. It follows that $\operatorname{des} \sigma = \operatorname{des} Q$ and $\operatorname{maj} \sigma = \operatorname{maj} Q$. So we can rewrite (12) as

$$M_n(q, t) = \sum_{(P,Q) \in \mathcal{H}_n} q^{\operatorname{maj} Q} t^{\operatorname{des} Q}.$$

We will need a lemma about what happens if we restrict this sum to the parallelogram polyominoes $\mathcal{P}_n \subseteq \mathcal{H}_n$.

Lemma 6.1. *We have*

$$\sum_{(P,Q)\in\mathcal{P}_n} q^{\operatorname{maj} Q} t^{\operatorname{des} Q} = M_{n-1}(q,t) + (q^{n-1}t-1)M_{n-2}(q,t).$$

Proof. Let $(P',Q') \in \mathcal{H}_{n-1}$ be obtained from $(P,Q) \in \mathcal{P}_n$ by shortening. Also let us write $P = p_1 \dots p_{n-1}E$ and $Q = q_1 \dots q_{n-1}N$. We have two cases.

If $q_{n-1} = E$, then $n-1 \in \operatorname{Des} Q$ which implies

$$q^{\operatorname{maj} Q} t^{\operatorname{des} Q} = q^{n-1+\operatorname{maj} Q'} t^{1+\operatorname{des} Q'} = q^{n-1} t q^{\operatorname{maj} Q'} t^{\operatorname{des} Q'}$$

Furthermore, P' and Q' both end with an E step and removal of that common step leaves a polyomino in $\mathcal{H}_{n-2}$. It follows that the (P,Q) in this case contribute $q^{n-1}tM_{n-2}(q,t)$ to the sum in the lemma.

For the second case we have $q_{n-1} = N$. It follows that $(P',Q') \in \mathcal{H}_{n-1}$ where the only restriction is that Q' end with a north step. In other words, we want the generating function for all polyominoes in $\mathcal{H}_{n-1}$ except for those whose lower path ends with an E step (which must coincide with the last step of the upper path which is always E). This is clearly $M_{n-1}(q,t) - M_{n-2}(q,t)$, and adding the contributions of the two cases we are done. □

Theorem 6.2. *For $n \geq 1$ we have*

$$M_n(q,t) = M_{n-1}(q,qt) + \sum_{k=2}^{n} \left[M_{k-1}(q,t) + (q^{k-1}t-1)M_{k-2}(q,t)\right] M_{n-k}(q,q^k t),$$

and

$$M_n(q,t) = M_{n-1}(q,t) + \sum_{k=0}^{n-2} M_k(q,t) \left[M_{n-k-1}(q,q^k t) + (q^{n-1}t-1)M_{n-k-2}(q,q^k t)\right].$$

Proof. To obtain the first equation, suppose $(P,Q) \in \mathcal{H}_n$. If both P and Q start with an E step then the generating function for such pairs is $M_{n-1}(q,qt)$ since each descent of Q is moved over one position.

Since Q always starts with an E-step, the only other possibility is for P to start with an N-step. Let z be the first point of intersection of P and Q after their initial vertex. Let Q_0 and Q_1 denote the portions of Q before and after z, respectively, and similarly for P_0 and P_1. Let $k = |Q_0| = |P_0|$. But then P_0 and Q_0 do not intersect between their initial point and z. Thus $(P_0,Q_0) \in \mathcal{P}_k$ and, from the previous lemma, the generating function for such pairs is the first factor in the summation.

We also have $|Q_1| = |P_1| = n-k$ and $(P_1,Q_1) \in \mathcal{H}_{n-k}$. Since Q_1 is preceded by a path with k steps, each of its descents will be increased by k. So the generating function for such pairs is $M_{n-k}(q,q^k t)$. Putting all the pieces together results in the first formula in the statement of the theorem.

To obtain the second, merely replace z in the proof just given by the last point of intersection of P and Q before their final vertex. □

We end this section with the observation that there is a close connection between the maps Γ and Υ defined in Sections 4 and 5, respectively. Specifically, the inverse of $\Gamma : \mathrm{Av}_n(321) \to \mathcal{D}_n$ coincides with the composition of the bijection from $\mathcal{D}_n$ to $\mathcal{H}_n$ used in [11] to prove Theorem 5.1 with the bijection $\Upsilon : \mathcal{H}_n \to \mathrm{Av}_n(321)$. So one can use Γ in place of Υ in some of the proofs. For example, the equation $I_n(q,t) = q^{n-1}tC_n(q, 1/q; t/q)$ which appears in the second proof of Theorem 1.1 can also be obtained from Γ as follows. If $\pi \in \mathrm{Av}_n(321)$ and $\Gamma(\pi) = P$, then by Proposition 4.1 and the definitions in Section 3, we have

$$\begin{aligned}
\mathrm{inv}(\pi) &= \mathrm{spea}\, P = n + \alpha(P) - \beta(P) - \mathrm{npea}\, P = n - 1 + \alpha(P) - \beta(P) - \mathrm{des}\, P,\\
\mathrm{lrm}(\pi) &= \mathrm{npea}\, P = 1 + \mathrm{des}\, P.
\end{aligned}$$

Thus,

$$I_n(q,t) = \sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}\, P} t^{\mathrm{npea}\, P} = q^{n-1}tC_n(q, 1/q; t/q).$$

Theorem 6.2 can also be proved using the bijection Γ. Note that each descent in a permutation $\pi \in \mathrm{Av}_n(321)$ corresponds to an occurrence of the string NEE in the Dyck path $\Gamma(\pi)$. Thus the statistics des and maj in π correspond to the number of occurrences and the sum of the x-coordinates after the first E in each occurrence of NEE in $\Gamma(\pi)$, respectively. Using the standard decomposition of a non-empty Dyck path as $P = NQER$ where Q and R are Dyck paths, as well as its reversal, we can keep track of these two statistics to obtain the recursions in Theorem 6.2.

7 Symmetry, unimodality, and mod 2 behavior of $M_n(q,t)$

The coefficients of the polynomials $M_n(q,t)$ have various nice properties which we now investigate. If $f(x) = \sum_k a_k x^k$ is a polynomial in x then we will use the notation

$$\begin{aligned}
[x^k]f(x) &= \text{coefficient of } x^k \text{ in } f(x)\\
&= a_k.
\end{aligned}$$

Our main object of study in this section will be the polynomial

$$A_{n,k}(q) = [t^k]M_n(q,t).$$

In other words, $A_{n,k}(q)$ is the generating function for the maj statistic over $\sigma \in \mathrm{Av}_n(321)$ having exactly k descents.

The first property which will concern us is symmetry. Consider a polynomial

$$f(x) = \sum_{i=r}^{s} a_i x^i$$

where $a_r, a_s \neq 0$. Call $f(x)$ *symmetric* if $a_i = a_j$ whenever $i + j = r + s$.

Theorem 7.1. *The polynomial $A_{n,k}(q)$ is symmetric for all n, k.*

Proof. If σ is counted by $A_{n,k}(q)$ then $\operatorname{des}\sigma = k$. Since σ avoids 321, it can not have two consecutive descents and so the minimum value of k is

$$1 + 3 + \cdots + (2k-1) = k^2$$

and the maximum value is

$$(n-1) + (n-3) + \cdots + (n-2k+1) = nk - k^2.$$

So it suffices to show that for $0 \le i \le nk$ we have $a_i = a_{nk-i}$ where

$$A_{n,k}(q) = \sum_i a_i q^i.$$

Let $\mathcal{A}_i$ be the permutations counted by a_i and let R_{180} denote rotation of the diagram of σ by 180 degrees. We claim R_{180} is a bijection between $\mathcal{A}_i$ and $\mathcal{A}_{nk-i}$ which will complete the proof. First of all, $R_{180}(321) = 321$ and so σ avoids 321 if and only if $R_{180}(\sigma)$ does so as well. If $\sigma \in \mathcal{A}_i$ then let $\operatorname{Des}\sigma = \{d_1, \ldots, d_k\}$ where $\sum_j d_j = i$. It is easy to see that $\operatorname{Des} R_{180}(\sigma) = \{n-d_1, \ldots, n-d_k\}$. It follows that $\operatorname{maj} R_{180}(\sigma) = nk - i$ and so $R_{180}(\sigma) \in \mathcal{A}_{nk-i}$. Thus R_{180} restricts to a well defined map from $\mathcal{A}_i$ to $\mathcal{A}_{nk-i}$. Since it is its own inverse, it is also a bijection. $\square$

Two other properties often studied for polynomials are unimodality and log concavity. The polynomial $f(x) = \sum_{i=0}^{s} a_i x^i$ is *unimodal* if there is an index r such that $a_0 \le \ldots \le a_r \ge \ldots \ge a_s$. It is *log concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $0 < i < s$. If all the a_i are positive, then log concavity implies unimodality.

Conjecture 7.2. *The polynomial $A_{n,k}(q)$ is unimodal for all n, k.*

This conjecture has been checked by computer for all $k < n \le 10$. The corresponding conjecture for log concavity is false, in particular, $A_{6,2}$ is not log concave.

The number-theoretic properties of the Catalan numbers have attracted some interest. Alter and Kubota [1] determined the highest power of a prime p dividing C_n using arithmetic means. Deutsch and Sagan [4] gave a proof of this result using group actions for the special case $p = 2$. Just considering parity, one gets the nice result that C_n is odd if and only if $n = 2^m - 1$ for some nonnegative integer m. Dokos et al. proved the following refinement of the "if" direction of this statement.

Theorem 7.3 (Dokos et al. [5])**.** *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[q^k]I_n(q,1) = \begin{cases} 1 & \text{if } k = 0 \\ \text{an even integer} & \text{if } k \ge 1. \end{cases} \qquad \square$$

In the same paper, the following statement was made as a conjecture which has now been proved by Killpatrick.

Theorem 7.4 (Killpatrick [13])**.** *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[q^k]M_n(q,1) = \begin{cases} 1 & \text{if } k = 0 \\ \text{an even integer} & \text{if } k \ge 1. \end{cases} \qquad \square$$

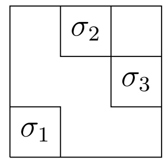


Figure 6: The diagram of $132[\sigma_1, \sigma_2, \sigma_3]$

We wish to prove a third theorem of this type. To do so, we will need the notion inflation for permutations. Given a permutation $\pi = a_1 \ldots a_n \in \mathfrak{S}_n$ and permutations $\sigma_1, \ldots, \sigma_n$, the *inflation* of π by the σ_i, written $\pi[\sigma_1, \ldots, \sigma_n]$, is the permutation whose diagram is obtained from the diagram of π by replacing the dot (i, a_i) by a copy of σ_i for $1 \le i \le n$. By way of example, Figure 6 shows a schematic diagram of an inflation of the form $132[\sigma_1, \sigma_2, \sigma_3]$. More specifically, $132[21, 1, 312] = 216534$.

Theorem 7.5. *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[t^k]M_n(1,t) = A_{n,k}(1) = \begin{cases} 1 & \text{if } k = 0, \\ \text{an even integer} & \text{if } k \ge 1. \end{cases}$$

Proof. We have $A_{n,0}(q) = 1$ since $\sigma = 12 \ldots n$ is the only permutation without descents and it avoids 321.

If $k \ge 1$, and $A_{n,k}(q)$ has an even number of terms then $A_{n,k}(1)$ must be even because it is a symmetric polynomial by Theorem 7.1. By the same token, if $A_{n,k}(q)$ has an odd number of terms, then $A_{n,k}(1)$ has the same parity as its middle term. Now consider R_{180} acting on the elements of $\mathcal{A}_{nk/2}$ as in the previous proof. Note that since n is odd, k must be even. Furthermore, this action partitions $\mathcal{A}_{nk/2}$ into orbits of size one and two. So it suffices to show that there are an even number of fixed points. If σ is fixed then its diagram must contain the center, c, of the square since this is a fixed point of R_{180}. Also, the NW and SE quadrants of σ with respect to c must be empty, since otherwise they both must contain dots (as one is taken to the other by R_{180}) and together with c this forms a 321. For the same reason, the SW quadrant of σ determines the NE one. Thus, the fixed points are exactly the inflations of the form $\sigma = 123[\tau, 1, R_{180}(\tau)]$ where $\tau \in \mathrm{Av}_{2^{m-1}-1}(321)$ has $k/2$ descents. By induction on m we have that the number of such τ, and hence the number of such σ, is even. $\square$

8 A refinement of Theorem 1.1 using continued fractions

We will now use a modification of the bijection in Section 2 together with the theory of continued fractions to give a third proof of Theorem 1.1. In fact, we will be able to keep track of a third statistic on permutations σ, namely

$$\operatorname{fix}\sigma = \text{the number of fixed points of } \sigma.$$

So consider the following polynomial

$$I_n(q,t,x) = \sum_{\sigma \in \mathrm{Av}_n(321)} q^{\operatorname{inv}\sigma} t^{\operatorname{lrm}\sigma} x^{\operatorname{fix}\sigma}$$

and the generating function

$$\mathfrak{I}(q,t,x;z) = \sum_{n\geq 0} I_n(q,t,x) z^n.$$

It is worth noting that

$$I_n(q,t,x/t) = \sum_{\sigma\in \mathrm{Av}_n(321)} q^{\mathrm{inv}\,\sigma} t^{\mathrm{exc}\,\sigma} x^{\mathrm{fix}\,\sigma} \tag{13}$$

where $\mathrm{exc}\,\sigma$ is the number of excedances of σ (i.e., the number of indices i such that $\sigma(i) > i$). This follows from the following fact.

Lemma 8.1. *Suppose $\sigma = a_1 a_2 \cdots a_n \in \mathrm{Av}_n(321)$. Then a_i is a left-right maximum if and only if $a_i \geq i$. Consequently,* $\mathrm{lrm}\,\sigma = \mathrm{exc}\,\sigma + \mathrm{fix}\,\sigma$.

Proof. If a_i is a left-right maximum, then a_i is greater than the $i-1$ elements to its left in σ, so $a_i > i-1$. Conversely, if a_i is not a left-right maximum then it is smaller than some element to its left in σ. Also, by Lemma 2.1(a), it is smaller than all $n-i$ elements to its right in σ. This implies that $n - a_i \geq n - i + 1$, whence $a_i \leq i - 1$. □

Continued fractions are very useful for enumerating weighted Motzkin paths M. If s is a step of M then we defined its *height* to be

$$\mathrm{ht}(s) = \text{the } y\text{-coordinate of the initial lattice point of } s.$$

If $\mathrm{ht}(s) = i$ then we assign s a weight $\mathrm{wt}\, s = u_i$, d_i, or l_i corresponding to s being an up, down, or level step, respectively. Weight paths $M = s_1 \ldots s_n$ and the set $\mathcal{M}_n$ of all such Motzkin paths by

$$\mathrm{wt}\, M = \prod_{j=1}^{n} \mathrm{wt}\, s_j$$

and

$$\mathrm{wt}\, \mathcal{M}_n = \sum_{M\in\mathcal{M}_n} \mathrm{wt}\, M.$$

For example, taking the path M in Figure 1 would give (ignoring the subscripts on the L's) $\mathrm{wt}\, M = u_0 l_1^2 u_1 l_2 d_2 l_1 d_1$.

In the sequel, we will use the following notation for continued fractions

$$F = \frac{a_1|}{|b_1} \pm \frac{a_2|}{|b_2} \pm \frac{a_3|}{|b_3} \pm \cdots \quad = \quad \cfrac{a_1}{b_1 \pm \cfrac{a_2}{b_2 \pm \cfrac{a_3}{b_3 \pm \cdots}}}. \tag{14}$$

We can now state Flajolet's classic result connecting continued fractions and weighted Motzkin paths.

Theorem 8.2 (Flajolet [10])**.** *If z is an indeterminate then*

$$\sum_{n\geq 0} \mathrm{wt}\, \mathcal{M}_n \; z^n = \frac{1|}{|1 - l_0 z} - \frac{u_0 d_1 z^2|}{|1 - l_1 z} - \frac{u_1 d_2 z^2|}{|1 - l_2 z} - \frac{u_2 d_3 z^2|}{|1 - l_3 z} - \cdots \tag{15}$$

is the generating function for weighted Motzkin paths. □

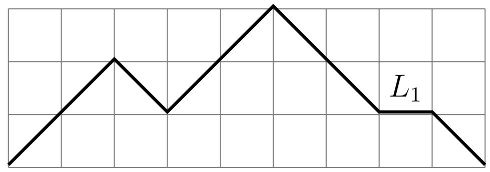


Figure 7: The Motzkin path $\nu(\sigma)$ associated with $\sigma = 361782495$

We will also use the same classification of continued fractions employed by Flajolet [10] which defines a Jacobi-type continued fraction as a continued fraction of the form

$$J(z) = \frac{1|}{|1 - b_0 z} - \frac{\lambda_1 z^2|}{|1 - b_1 z} - \frac{\lambda_2 z^2|}{|1 - b_2 z} - \frac{\lambda_3 z^2|}{|1 - b_3 z} - \cdots$$

and a Stieljes-type continued fraction as a continued fraction of the form

$$S(z) = \frac{1|}{|1} - \frac{\lambda_1 z|}{|1} - \frac{\lambda_2 z|}{|1} - \frac{\lambda_3 z|}{|1} - \cdots .$$

Following the book of Jones and Thron [12], a continued fraction of the form

$$T(z) = \frac{1|}{|1 - b_0 z} - \frac{\lambda_1 z|}{|1 - b_1 z} - \frac{\lambda_2 z|}{|1 - b_2 z} - \frac{\lambda_3 z|}{|1 - b_3 z} - \cdots .$$

will be called a Thron-type continued fraction.

In order to derive a continued fraction expansion for $\mathfrak{I}(q,t,x;z)$, we will set up a bijection between $\mathrm{Av}_n(321)$ and a subset of $\mathcal{M}_n^{(2)}$. Call $M \in \mathcal{M}_n^{(2)}$ *restricted* if it has no L_0 steps at height 0. Let $\mathcal{R}_n$ be the set of such paths. In the following proof, we will use the same definitions and notation as in Section 2.

Theorem 8.3. *The series $\mathfrak{I}(q,t,x;z)$ has Jacobi-type continued fraction expansion*

$$\mathfrak{I}(q,t,x;z) = \frac{1|}{|1 - txz} - \frac{tqz^2|}{|1 - (1+t)qz} - \frac{tq^3 z^2|}{|1 - (1+t)q^2 z} - \frac{tq^5 z^2|}{|1 - (1+t)q^3 z} - \cdots .$$

Proof. We define a bijection $\nu : \mathrm{Av}_n(321) \to \mathcal{R}_n$ in a way similar to the bijection μ in the first proof of Theorem 1.1, but without the shift. Specifically, given $\sigma = a_1 \dots a_n \in \mathrm{Av}_n(321)$ with $\operatorname{val}\sigma = (v_1, \dots, v_n)$ and $\operatorname{pos}\sigma = (p_1, \dots, p_n)$, we let $\nu(\sigma) = M = s_1 \dots s_n$ where

$$s_i = \begin{cases} U & \text{if } v_i = 0 \text{ and } p_i = 1, \\ D & \text{if } v_i = 1 \text{ and } p_i = 0, \\ L_0 & \text{if } v_i = p_i = 0, \\ L_1 & \text{if } v_i = p_i = 1. \end{cases}$$

Continuing the example from the beginning of the paper, $\sigma = 361782495$ would be mapped to the path in Figure 7.

We must show that ν is well defined in that $M \in \mathcal{R}_n$. Defining the inverse map and proving it is well defined is similar and so left to the reader. The fact that M is a Motzkin path follows because, by Lemma 2.1(b), in every prefix of $\operatorname{pos}\sigma$ the number of ones is at least as great as the number in the corresponding prefix of $\operatorname{val}\sigma$, with equality for all of σ. This forces similar inequalities and equality between the number of up-steps and the number of down-steps in M. Thus M stays weakly above the x-axis and ends on it.

To see that M has no L_0-steps on the x-axis, note first that all steps before the first U-step (if any) must be of the form L_1 because, if not, then the index i of the first such L_0-step would contradict Lemma 2.1(b). Also, any time M returns to the x-axis, it must be with $s_j = D$ for some j. So the corresponding prefixes of $\operatorname{val}\sigma$ and $\operatorname{pos}\sigma$ have the same number of ones and this implies that $a_1 \ldots a_j$ are $1, \ldots, j$ in some order. Now using an argument similar to the one just given, one sees that there can be no L_0-step before the next U-step. This completes the proof that μ is well defined.

We now claim that

$$\begin{aligned}
\operatorname{fix}\sigma &= \text{the number of } L_1\text{-steps at height } 0, && (16)\\
\operatorname{lrm}\sigma &= \#U(M) + \#L_1(M), && (17)\\
\operatorname{inv}\sigma &= \operatorname{area} M. && (18)
\end{aligned}$$

Let us prove the first equation. If $s_j = L_1$ with $\operatorname{ht}(s_j) = 0$ then, as in the proof that ν is well defined, $a_1 \ldots a_{j-1}$ are the numbers $1, \ldots, j-1$ in some order. Thus if $v_j = p_j = 1$ then both the position and value of a_j correspond to a left-right maximum. This forces $a_j = j$ and so we have a fixed point. Similar considerations show that every fixed point in a 321-avoiding permutation is a left-right maximum corresponding to an L_1 step at height 0.

Equation (17) follows immediately from the fact that the number of left-right maxima in σ equals the number of ones in $\operatorname{pos}\sigma$, and the corresponding steps in M are of the form U or L_1.

For the final equality, first recall that all inversions of σ are between a left-right maximum m and a non-left-right maximum to its right by Lemma 2.1(a). So if m is in position p then, because everything to its left is smaller, it creates $m - p$ inversions. Also, the maximum values and their positions in σ are given by iv_i and jp_j, respectively, whenever $v_i, p_j = 1$. Since $iv_i = jp_j = 0$ whenever $v_i, p_j = 0$ we have

$$\operatorname{inv}\sigma = \sum_{v_i=1} iv_i - \sum_{p_j=1} jp_j = \sum_{i=1}^{n} (v_i - p_i)i.$$

As far as the area, we start by noting that $\operatorname{area} M = \sum_j \operatorname{ht} s_j$. Furthermore, $\operatorname{ht}(s_j)$ is just the difference between the number of up-steps and down-steps preceding s_j. For any step s_i, we have $p_i - v_i = 1, -1$, or 0 corresponding to s_i being an up-, down-, or level-step, respectively. So $\operatorname{ht}(s_j) = \sum_{i<j}(p_i - v_i)$. Combining expressions, interchanging summations, and using the fact that $\operatorname{val}\sigma$ and $\operatorname{pos}\sigma$ have the same number of ones, gives

$$\operatorname{area} M = \sum_{j=1}^{n} \sum_{i<j} (p_i - v_i) = \sum_{i=1}^{n} [(n-i)p_i - (n-i)v_i] = \sum_{i=1}^{n} (v_i - p_i)i.$$

Comparing this expression with the one derived for $\operatorname{inv}\sigma$ in the previous paragraph completes the proof of (18).

To finish the demonstration of the theorem, we just need to set the weights in Theorem 8.2 in light of (16)–(18). The only level steps at height 0 are L_1 which contribute to both $\operatorname{fix}\sigma$ and $\operatorname{lrm}\sigma$. So we let $l_0 = tx$. At heights $h \geq 1$ we have both L_0 and L_1 steps. The former only contribute to inv by adding h, while the later also increase the lrm statistic by one, so we have $l_h = (1+t)q^h$. Similar reasoning gives $u_h = tq^h$ and $d_h = q^h$ which, after plugging into equation (15), completes the proof. $\square$

The following refinement of Theorem 1.1 is a simple consequence of the preceding result.

Theorem 8.4. *For $n \geq 1$,*

$$I_n(q,t,x) = txI_{n-1}(q,t,x) + \sum_{k=0}^{n-2} q^{k+1} I_k(q,t,1)\left[I_{n-1-k}(q,t,x) - t(x-1)I_{n-2-k}(q,t,x)\right].$$

Proof. We can derive from the continued fraction expansion of $\mathfrak{I}(q,t,x;z)$ in Theorem 8.3 that

$$\mathfrak{I}(q,t,x;z) = \cfrac{1}{1 - txz - \cfrac{tqz^2}{-qz + \cfrac{1}{\mathfrak{I}(q,t,1;qz)}}}.$$

After simplification, this leads to the functional equation

$$\mathfrak{I}(q,t,x;z) = 1 + txz\mathfrak{I}(q,t,x;z) + qz\mathfrak{I}(q,t,1;qz)\left[\mathfrak{I}(q,t,x;z) - 1 - tz(x-1)\mathfrak{I}(q,t,x;z)\right]. \quad (19)$$

Extracting the coefficient of z^n on both sides gives the desired recursion. $\square$

We note that the $q = 1$ case of the functional equation (19) is, by (13), equivalent to equation (1) in [6]. It can be explicitly solved as done in equation (2) of the work just cited.

One can simplify the continued fraction in Theorem 8.3 in the case $x = 1$. The *nth convergent* of the continued fraction (14) is

$$c_n(F) = \frac{a_1|}{|b_1} \pm \frac{a_2|}{|b_2} \pm \cdots \pm \frac{a_n|}{|b_n}.$$

We wish to construct continued fractions, called the *even* and *odd parts* of F and denoted F_e and F_o, such that $c_{2n}(F) = c_n(F_e)$ and $c_{2n+1}(F) = c_n(F_o)$, respectively. The following theorem shows how to do this when $b_n = 1$ for all n.

Theorem 8.5 ([12]). *We have*

$$\begin{aligned} \frac{a_1|}{|1} + \frac{a_2|}{|1} + \frac{a_3|}{|1} + \cdots &= \frac{a_1|}{|1+a_2} - \frac{a_2a_3|}{|1+a_3+a_4} - \frac{a_4a_5|}{|1+a_5+a_6} - \frac{a_6a_7|}{|1+a_7+a_8} + \cdots \\ &= a_1 - \frac{a_1a_2|}{|1+a_2+a_3} - \frac{a_3a_4|}{|1+a_4+a_5} - \frac{a_5a_6|}{|1+a_6+a_7} - \frac{a_7a_8|}{|1+a_8+a_9} - \cdots \end{aligned}$$

as even and odd parts, respectively, of the first continued fraction. $\square$

Using the concept of even and odd parts, we see that there is a simple and well-known relationship between Stieljes- and Jacobi-type continued fractions [12, p. 129]. More precisely, taking the even part of a Stieljes-type continued fraction gives

$$\frac{1|}{|1} - \frac{\lambda_1 z|}{|1} - \frac{\lambda_2 z|}{|1} - \cdots = \frac{1|}{|1-\lambda_1 z} - \frac{\lambda_1\lambda_2 z^2|}{|1-(\lambda_2+\lambda_3)z} - \frac{\lambda_3\lambda_4 z^2|}{|1-(\lambda_4+\lambda_5)z} - \frac{\lambda_5\lambda_6 z^2|}{|1-(\lambda_6+\lambda_7)z} - \cdots$$

where the right-hand side is of Jacobi type. Combining Theorem 8.3 when $x = 1$ with the above equation, we get the following result.

Corollary 8.6. *Set* $\mathfrak{I}(q,t;z) := \mathfrak{I}(q,t,1;z)$. *The generating function* $\mathfrak{I}(q,t;z)$ *has the Stieltjes-type continued fraction expansion*

$$\mathfrak{I}(q,t;z) = \frac{1|}{|1} - \frac{tz|}{|1} - \frac{qz|}{|1} - \frac{tqz|}{|1} - \frac{q^2z|}{|1} - \frac{tq^2z|}{|1} - \frac{q^3z|}{|1} - \frac{tq^3z|}{|1} - \frac{q^4z|}{|1} - \frac{tq^4z|}{|1} - \cdots \tag{20}$$

It is interesting to note that there is a second recursion for $I_n(q,t)$ which follows from a result of Krattenthaler.

Theorem 8.7 (Krattenthaler [14])**.** *We have*

$$\mathfrak{I}(q,t;z) = \frac{1|}{|1-(t-1)z} - \frac{z|}{|1-(tq-1)z} - \frac{z|}{|1-(tq^2-1)z} - \frac{z|}{|1-(tq^3-1)z} - \cdots$$

as the Thron-type continued fraction expansion of $\mathfrak{I}(q,t;z)$. □

Corollary 8.8. *For* $n \geq 1$,

$$I_n(q,t) = tI_{n-1}(q,t) + \sum_{k=0}^{n-2} I_k(q,t)I_{n-1-k}(q,qt).$$

Proof. Simple manipulation of the continued fraction in Krattenthaler's Theorem gives the functional equation

$$\mathfrak{I}(q,t;z) = 1 + (t-1)z\mathfrak{I}(q,t;z) + z\mathfrak{I}(q,t;z)\mathfrak{I}(q,qt;z).$$

Taking the coefficient of z^n on both sides of this equation finishes the proof. □

It is worth noting that, *a priori*, it is not at all clear that the recursions in Theorem 1.1 and the above corollary generate the same sequence of polynomials. The relationship between these two recursions can be interpreted in terms of the statistics sumpeaks and sumtunnels, introduced in Section 4, as follows. Using the standard decomposition of non-empty Dyck paths as $P = NQER$, where Q and R are Dyck paths, the generating function for $\mathcal{D}_n$ with respect to the statistics $(\mathrm{spea}, \mathrm{npea})$ satisfies the recursion in Corollary 8.8. On the other hand, using the same decomposition, the generating function for $\mathcal{D}_n$ with respect to the statistics $(\mathrm{stun}, n-\mathrm{des})$ satisfies the recursion in Theorem 1.1. Thus, Theorem 4.2 implies that the two recursions are equivalent.

9 Refined sign-enumeration of 321-avoiding permutations

Simion and Schmidt [17] considered the signed enumeration of various permutation classes of the form $\sum_{\sigma\in \mathrm{Av}_n(\pi)}(-1)^{\mathrm{inv}\,\sigma}$. In this section we will rederive their theorem for $\mathrm{Av}_n(321)$ using the results of the previous section. In addition, we will provide a more refined signed enumeration which also keeps track of the lrm statistic. We should note that Reifegerste [15] also has a refinement which takes into account the length of the longest increasing subsequence of σ.

Let C_n be the nth Catalan number and consider the generating function $C(z) = \sum_{n\geq 0} C_n z^n$. It is well known that $C(z)$ satisfies the functional equation $C(z) = 1 + zC(z)^2$. Rewriting this as $C(z) = 1/(1 - zC(z))$ and iteratively substituting for $C(z)$, we obtain the also well-known continued fraction

$$C(z) = \frac{1|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \cdots .$$

Now plug $q = -1$ and $t = 1$ into the continued fraction (20) to obtain

$$\mathfrak{I}(-1,1;z) = \frac{1|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} \cdots .$$

Using Theorem 8.5 to extract the odd part of this expansion gives

$$\mathfrak{I}(-1,1;z) = 1 + \frac{z|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \cdots .$$

Comparing this to the continued fraction for $C(z)$, we see that

$$\mathfrak{I}(-1,1;z) = 1 + zC(z^2).$$

Taking the coefficient of z^n on both sides yields the following result.

Theorem 9.1 (Simion and Schmidt [17])**.** *For all $n \geq 1$, we have*

$$I_{2n}(-1,1) = \sum_{\sigma\in \mathrm{Av}_{2n}(321)} (-1)^{\mathrm{inv}\,\sigma} = 0 \quad \textit{and} \quad I_{2n+1}(-1,1) = \sum_{\sigma\in \mathrm{Av}_{2n+1}(321)} (-1)^{\mathrm{inv}\,\sigma} = C_n.$$

□

Since our refined sign-enumeration will involve the parameter lrm, we recall (but will not use) the folklore result that the enumerating polynomial of $\mathrm{Av}_n(321)$ according to the lrm statistic is the nth *Narayana polynomial*, i.e.,

$$I_n(1,t) = \sum_{\sigma\in \mathrm{Av}_n(321)} t^{\mathrm{lrm}\,\sigma} = \sum_{k=1}^{n} N_{n,k} t^k,$$

where the *Narayana number* $N_{n,k}$ is given by $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n \geq k \geq 1$.

Theorem 9.2. *For all $n \geq 1$,*

$$I_n(-1,t) = \sum_{\sigma\in \mathrm{Av}_n(321)} (-1)^{\mathrm{inv}\,\sigma} t^{\mathrm{lrm}\,\sigma} = \sum_{k=1}^{n} (-1)^{n-k} s_{n,k} t^k \qquad (21)$$

where $s_{n,k}$ is defined for $n \geq k \geq 1$ by

$$s_{n,k} = \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} \binom{\lceil \frac{n-1}{2} \rceil}{\lceil \frac{k-1}{2} \rceil}.$$

Moreover,

$$I_{2n}(-1,t) = (t-1) I_{2n-1}(-1,t), \tag{22}$$

$$(n+1) I_{2n+1}(-1,t) = 2((1+t^2)n - t) I_{2n-1}(-1,t) - (1-t^2)^2 (n-1) I_{2n-3}(-1,t). \tag{23}$$

Proof. Let $\mathfrak{I}(t;z)$ and $\mathfrak{I}_{odd}(t;z)$ be the power series defined as

$$\mathfrak{I}(t;z) = \sum_{n \geq 0} I_n(-1,t) z^n \quad \text{and} \quad \mathfrak{I}_{odd}(t;z) = \sum_{n \geq 0} I_{2n+1}(-1,t) z^n.$$

By equation (20), we have

$$\begin{aligned} \mathfrak{I}(t;z) &= \frac{1|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \frac{z|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \frac{z|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \cdots \\ &= \cfrac{1}{1 - \cfrac{tz}{1 + \cfrac{z}{1 + \cfrac{tz}{1 - z\mathfrak{I}(t;z)}}}} \end{aligned}$$

which, after simplification, leads to the functional equation

$$(1+z-tz)z\, \mathfrak{I}(t;z)^2 - (1+2z+z^2-t^2z^2)\, \mathfrak{I}(t;z) + (1+z+tz) = 0.$$

Solving this quadratic equation, we obtain

$$\mathfrak{I}(t;z) = \frac{1+2z+(1-t^2)z^2 - \sqrt{1-2(1+t^2)z^2+(1-t^2)^2z^4}}{2z(1+z-tz)}. \tag{24}$$

Noticing that $\mathfrak{I}_{odd}(t;z) = (\mathfrak{I}(t;\sqrt{z}) - \mathfrak{I}(t;-\sqrt{z}))/2\sqrt{z}$ and using (24), we obtain after a routine computation

$$\mathfrak{I}_{odd}(t;z) = \frac{1-(1-t)^2z - \sqrt{1-2(1+t^2)z+(1-t^2)^2z^2}}{2z(1-(1-t)^2z)}. \tag{25}$$

It follows from (24) that

$$(1+z-tz)\mathfrak{I}(t;z) + (1-z+tz)\mathfrak{I}(t;-z) = 2.$$

Extracting the coefficient of z^{2n} on both sides of the last equality, we obtain (22). Using (25), it is easily checked that $\mathfrak{I}_{odd}$ satisfies the differential equation

$$z\big(1-2(1+t^2)z+(1-t^2)^2z^2\big)\, \mathfrak{I}'_{odd}(t;z) + \big(1-2(1-t+t^2)z+(1-t^2)^2z^2)\big)\, \mathfrak{I}_{odd}(t;z) - t = 0,$$

where $\mathfrak{I}'_{odd}(t;z)$ is the derivative with respect to z. Extracting the coefficient of z^n on both sides of the last equality, we obtain (23).

We now turn our attention to (21). Clearly, we have

$$[t^{2k+1}]I_{2n+1}(-1,t) = [t^k z^n]\frac{\mathfrak{I}_{odd}(\sqrt{t};z) - \mathfrak{I}_{odd}(-\sqrt{t};z)}{2\sqrt{t}} = [t^k][z^n]\frac{1}{\sqrt{1-2(1+t)z+(1-t)^2z^2}} \quad (26)$$

where the last equality follows from (25). Using the Lagrange inversion formula, we can show that

$$[z^n]\frac{1}{\sqrt{1-2(1+t)z+(1-t)^2z^2}} = [x^n](1+(1+t)x+tx^2)^n. \quad (27)$$

Combining (26) with the above relation, we obtain

$$[t^{2k+1}]I_{2n+1}(-1,t) = [x^n][t^k](1+(1+t)x+tx^2)^n = [x^n]\binom{n}{k}x^k(1+x)^n = \binom{n}{k}^2. \quad (28)$$

Similarly, we have

$$[t^{2k}]I_{2n+1}(-1,t) = [t^k z^n]\frac{\mathfrak{I}_{odd}(\sqrt{t};z) + \mathfrak{I}_{odd}(-\sqrt{t};z)}{2} = [t^k][z^n]\frac{1}{2z} - \frac{1-z-tz}{2z\sqrt{1-2(1+t)z+(1-t)^2z^2}}.$$

This, combined first with (26) and then (28), yields

$$[t^{2k}]I_{2n+1}(-1,t) = \frac{1}{2}\left([t^{2k+1}]I_{2n+1}(t,-1)+[t^{2k-1}]I_{2n+1}(t,-1)-[t^{2k+1}]I_{2n+3}(t,-1)\right) = -\binom{n}{k-1}\binom{n}{k}.$$

This proves that (21) is true when n is odd. Combining this with (22) shows that the formula also holds when n is even. □

To see why the previous result implies the one of Simion and Schmidt, plug $t=1$ into the equations for $I_{2n}(-1,t)$ and $I_{2n+1}(-1,t)$. In the former case we immediately get $I_{2n}(-1,1)=0$ because of the factor of $t-1$ on the right. In the latter, we get the equation $(n+1)I_{2n+1}(-1,1) = 2(2n-1)I_{2n-1}(-1,1)$. The fact that $I_{2n+1}(-1,1)=C_n$ now follows easily by induction.

Finally, it is interesting to note that the numbers $s_{n,k}$ which arise in the signed enumeration of $\mathrm{Av}_n(321)$ have a nice combinatorial interpretation. Recall that *symmetric* Dyck paths are those $P = s_1 \ldots s_{2n}$ which are the same read forwards as read backwards. The following result appears in Sloane's Encyclopedia [18]: For $n \geq k \geq 1$, the number $s_{n,k}$ is equal to the number of symmetric Dyck paths of semilength n with k peaks.

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