In this article, we assume $K$ is a non-archimedean local field.

1 Review: Invariant maps for unramified extensions

Recall that for a finite unramified extension of local fields $L/K$, the invariant map 

$$\text{inv}_{L/K} : H^2(\text{Gal}(L/K), L^\times) \to \mathbb{Q}/\mathbb{Z}$$

is defined to be the composition 

$$H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\text{inv}_{L/K}} H^2(\text{Gal}(L/K), \mathbb{Z}) \leftarrow H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$$f \mapsto f(\text{Frob}_{L/K})$$

where Frob$_{L/K}$ is the Frobenius element for $L/K$. Notice that the invariant map is injective and functorial in $L$: for $K \subset L \subset M$ finite unramified extensions, the following diagram commutes

$$
\begin{array}{ccc}
H^2(\text{Gal}(L/K), L^\times) & \xrightarrow{\text{inv}_{L/K}} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \text{Inf} & & \\
H^2(\text{Gal}(M/K), M^\times) & \xleftarrow{\text{inv}_{M/K}} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

**Theorem 1.1.** Let $K$ be a non-archimedean local field. There exists an isomorphism 

$$\text{inv}_K : H^2(\text{Gal}(K_{ur}/K), (K_{ur}^\times)^{\times}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

such that the composition with the inflation map induces an isomorphism

$$\text{inv}_{L/K} : H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\cong} \frac{1}{[L : K]}\mathbb{Z}/\mathbb{Z}$$

for every finite unramified extension $L/K$. 

1
2 Invariant maps for $K^{sep}/K$

In this section we will discuss the extension of the invariant map to a separable closure $K^{sep}$ of a non-archimedean local field $K$.

**Proposition 2.1.** Let $L/K$ be a finite separable extension of non-archimedean local fields. Then, there exists a homomorphism

$$\phi : H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times)$$

such that the diagram commutes

$$\begin{array}{ccc}
H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) & \xrightarrow{\text{inv}_K} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\phi} & & \downarrow{[L:K]} \\
H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times) & \xrightarrow{\text{inv}_L} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Furthermore, if $L/K$ is Galois, then the kernel $\ker(\phi)$ can be identified as a cyclic subgroup of order $[L : K]$ in $H^2(\text{Gal}(L/K), L^\times)$.

**Proof.** First suppose $L/K$ is a finite Galois extension. The Hilbert’s 90 tells us that $H^1(\text{Gal}(L^{ur}/L), (L^{ur})^\times) = 0$ and $H^1(\text{Gal}(L^{ur}/K^{ur}), (L^{ur})^\times) = 0$. Therefore, we have the following inflation-restriction sequences:

$$0 \to H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\text{Inf}} H^2(\text{Gal}(L^{ur}/K), (L^{ur})^\times) \xrightarrow{\text{Res}} H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times)$$

Moreover, if $L/K$ is a finite separable extension of non-archimedean local fields. Then, there exists a homomorphism

Define $\phi : H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times)$ as

$$\phi = \text{Res} \circ \text{Inf}'.$$

Now, assuming $\phi$ is the map in the first part of the proposition, we then have

$$\ker(\phi) \cong \frac{1}{[L : K]}\mathbb{Z}/\mathbb{Z}$$

under the invariant map. Note that

$$\text{Inf}'(\ker(\phi)) \subseteq \ker(\text{Res}) = \text{Im}(\text{Inf}) = H^2(\text{Gal}(L/K), L^\times)$$

and so $\ker(\phi)$ can be identified as a cyclic subgroup of order $[L : K]$ in $H^2(\text{Gal}(L/K), L^\times)$.

Now, we drop the Galois assumption and prove the first part of the proposition. Notice that the map $\phi = \text{Res} \circ \text{Inf}'$ still makes sense even when $L/K$ is not Galois. Recall that $[L : K] = ef$ where $e$ is the ramification index and $f$ is the inertia degree.

Writing out the definitions of $\text{inv}_K$ and $\text{inv}_L$:

$$\begin{array}{ccc}
H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) & \xrightarrow{\text{inv}_K} & H^2(\text{Gal}(K^{ur}/K), \mathbb{Z}) \\
\downarrow{\phi} & & \downarrow{e\phi} \\
H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times) & \xrightarrow{\text{inv}_L} & H^2(\text{Gal}(L^{ur}/L), \mathbb{Z})
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Q}/\mathbb{Z} & \xrightarrow{\delta} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\phi} & & \downarrow{e\phi} \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{\delta} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Q}/\mathbb{Z} & \xrightarrow{n=ef} & \mathbb{Q}/\mathbb{Z}
\end{array}$$
The first commutative square is given by the valuations and the fact that $v_{L\mid K} = ev_K$.

\[
\begin{array}{c}
(K^{ur})^\times \xrightarrow{v_K} \mathbb{Z} \\
\downarrow \\
(L^{ur})^\times \xrightarrow{v_L} \mathbb{Z}
\end{array}
\]

For the second commutative square, we have used the fact that $\phi$ is defined by the composition of inflation and restriction maps, both of which commute with the boundary maps.

For the last commutative square, note that by writing $L^{ur} = L \cdot K^{ur}$, we have an injection

\[
\text{Gal}(L^{ur}/L) \hookrightarrow \text{Gal}(K^{ur}/K)
\]

\[
\sigma \mapsto \sigma|_{K^{ur}}.
\]

In particular,

\[
\text{Frob}_L|_{K^{ur}} = \text{Frob}_K^f.
\]

So, for $g \in H^1(\text{Gal}(K^{ur}/K), \mathbb{Q}/\mathbb{Z})$,

\[
g|_{\text{Gal}(L^{ur}/L)}(\text{Frob}_L) = g(\text{Frob}_K^f) = f \cdot g(\text{Frob}_K).
\]

Now, we want to extend inv$_K$ to arbitrary separable extensions of $K$. For this we require what Neukirch calls the class field axiom:

**Theorem 2.2. (Class Field Axiom)** Let $L/K$ be a cyclic extension of non-archimedean local fields. Then,

\[
|H^k(\text{Gal}(L/K), L^\times)| = \begin{cases} [L : K], & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.
\]

**Corollary 2.3.** Let $L/K$ be a finite Galois extension of non-archimedean local fields. Then, $H^2(\text{Gal}(L/K), L^\times)$ is cyclic of order $[L : K]$.

**Proof.** We know from proposition 2.1 that $H^2(\text{Gal}(L/K), L^\times)$ contains a cyclic subgroup of order $[L : K]$ and so we need to show they are equal.

We prove this by induction on $[L : K]$. First note that if $L/K$ is cyclic, then the result follows from the class field axiom. Also, it is a fact from number theory that the Galois group of an extension of non-archimedean local fields is solvable. So by Galois correspondence, there exists a tower of $K \subsetneq E \subsetneq L$ of Galois extensions with

\[
|H^2(\text{Gal}(L/E), L^\times)| = [L : E] \\
|H^2(\text{Gal}(E/K), E^\times)| = [E : K].
\]

On the other hand, we have the inflation-restriction sequence

\[
0 \to H^2(\text{Gal}(E/K), E^\times) \xrightarrow{\text{Inf}} H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\text{Res}} H^2(\text{Gal}(L/E), L^\times).
\]
It follows that
\[ |H^2(\text{Gal}(L/K), L^\times)| \leq |H^2(\text{Gal}(L/E), L^\times)| \cdot |H^2(\text{Gal}(E/K), E^\times)| = [L : K]. \]

Now, we show that the invariant map extends to a separable closure \( K^{\text{sep}} \).

**Theorem 2.4.** Let \( K \) be a non-archimedean local field. There is an isomorphism

\[
\text{inv}_K : H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}
\]

such that for every finite Galois extension \( L/K \), the composition with the inflation map induces

\[
\text{inv}_{L/K} : H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\sim} \frac{1}{[L : K]}\mathbb{Z}/\mathbb{Z}
\]

which coincides with the invariant map when \( L/K \) is unramified. Moreover,

\[
0 \longrightarrow H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\text{Inf}} H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) \xrightarrow{\text{Res}} H^2(\text{Gal}(K^{\text{sep}}/L), (K^{\text{sep}})^\times) \xrightarrow{\text{Inv}} 0
\]

\[
0 \longrightarrow \frac{1}{[L : K]}\mathbb{Z}/\mathbb{Z} \xrightarrow{\text{Inv}} \mathbb{Q}/\mathbb{Z} \xrightarrow{[L : K]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{Inv}} 0.
\]

**Proof.** As a consequence of Hilbert’s 90, the inflation map

\[
\text{Inf} : H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(K^{sep}/K), (K^{sep})^\times)
\]

is injective. From proposition 2.1 and corollary 2.3 for all finite Galois extensions \( L/K \), under the homomorphism

\[
\phi : H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times),
\]

the kernel \( \ker(\phi) = H^2(\text{Gal}(L/K), L^\times) \) is a cyclic subgroup of order \( [L : K] \) in \( H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \). But since the cohomology group is the direct system of the system formed by the inflation maps between finite Galois group extensions of \( K \), that is

\[
H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) = \lim_{\text{L/K finite Galois}} H^2(\text{Gal}(L/K), L^\times),
\]

the inflation map \( \text{Inf} : H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) \) is an isomorphism. □
3 The fundamental class

**Definition 3.1.** Let \( L/K \) be a finite Galois extension of non-archimedean local fields. The **fundamental class** \( u_{L/K} \in H^2(\text{Gal}(L/K), L^\times) \) is the preimage of \( \frac{1}{[L:K]} \mathbb{Z}/\mathbb{Z} \) under the invariant map \( \text{inv}_{L/K} \).

We have some nice properties of the fundamental classes in a tower of Galois extensions.

**Lemma 3.2.** Suppose \( K \subseteq E \subseteq L \) is a tower of finite Galois extensions of non-archimedean local fields. Then,

1. \( \text{Res}(u_{L/K}) = u_{L/E} \).
2. \( \text{CoRes}(u_{L/E}) = [E : K]u_{L/K} \).
3. \( \text{Inf}(u_{E/K}) = [L : E]u_{L/K} \).

**Proof.** For part (c), recall that the inflation map \( \text{Inf} \) commutes with the invariant maps, that is

\[
H^2(\text{Gal}(E/K), E^\times) \xrightarrow{\text{Inf}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{inv}_{E/K}} H^2(\text{Gal}(L/K), L^\times).
\]

Since \( \text{inv}_{E/K}(u_{E/K}) = \frac{1}{[E:K]} \mathbb{Z}/\mathbb{Z} \) and \( \text{inv}_{L/K} = \frac{1}{[L:K]} \mathbb{Z}/\mathbb{Z} \), we have

\[
\text{Inf}(u_{E/K}) = [L : E]u_{L/K}.
\]

For part (a), we claim that

\[
H^2(\text{Gal}(L/K), L^\times) \xleftarrow{\text{Inf}} H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z}
\]

The second commutative square is given by theorem 2.4. We check that the induced map on \( H^2(\text{Gal}(L/K), L^\times) \) is the restriction map into \( H^2(\text{Gal}(L/E), L^\times) \). Notice the map the restriction map \( \text{Res} : H^2(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times) \) is induced by the inclusion \( \text{Gal}(K^{\text{sep}}/E) \hookrightarrow \text{Gal}(K^{\text{sep}}/E) \) whereas the inflation maps are given by quotient maps \( \text{Gal}(K^{\text{sep}}/K) \to \text{Gal}(L/K) \) and \( \text{Gal}(K^{\text{sep}}/E) \to \text{Gal}(L/E) \):

\[
\text{Gal}(L/K) = \text{Gal}(K^{\text{sep}}/K) / \text{Gal}(K^{\text{sep}}/L) \hookrightarrow \text{Gal}(K^{\text{sep}}/K)
\]
\[
\text{Gal}(L/E) = \text{Gal}(K^{\text{sep}}/E) / \text{Gal}(K^{\text{sep}}/L) \hookrightarrow \text{Gal}(K^{\text{sep}}/E)
\]

and hence the map \( i \) is an inclusion. This shows that the induced map on \( H^2(\text{Gal}(L/K), L^\times) \) is the restriction map into \( H^2(\text{Gal}(L/E), L^\times) \), which proves part (a).

For part (b), we have

\[
\text{CoRes}(u_{L/E}) = \text{CoRes}(\text{Res}(u_{L/K})) = [E : K]u_{L/K}.
\]

\qed
4 The Local Artin Map for $K^{ab}/K$

Our ultimate goal in this section is to define the local Artin map for $K^{ab}/K$. First, we define the local Artin map for a finite Galois extension $L/K$ of non-archimedean local fields. Recall that for $G = \text{Gal}(L/K)$, we have the isomorphisms

$$G^{ab} \xrightarrow{\cong} I_G/I_G^2 \overset{\delta_0}{\cong} H_1(G, \mathbb{Z}) =: \hat{H}^{-2}(G, \mathbb{Z})$$

where the second isomorphism is given by the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$ 

Furthermore, for any subgroup $H \subset G = \text{Gal}(L/K)$, we have $H^1(H, L^\times) = 0$ and $H^2(H, L^\times)$ is cyclic of order $|H|$. Hence, Tate’s theorem gives an isomorphism

$$\hat{H}^n(G, \mathbb{Z}) \xrightarrow{\Phi_{L/K}} \hat{H}^{n+2}(G, L^\times)$$

for a choice of generator of $\hat{H}^2(G, L^\times)$.

**Definition 4.1.** Let $L/K$ be a finite Galois extension of non-archimedean local fields with Galois group $\text{Gal}(L/K)$. The local Artin map

$$\theta_{L/K} : K^\times/N_{L/K}(L^\times) \xrightarrow{\cong} \text{Gal}(L/K)^{ab}$$

is the inverse of the isomorphisms

$$\text{Gal}(L/K)^{ab} \cong \hat{H}^{-2}(\text{Gal}(L/K), \mathbb{Z}) \xrightarrow{\Phi_{L/K}} \hat{H}^0(\text{Gal}(L/K), L^\times) =: K^\times/N_{L/K}(L^\times)$$

where $\Phi_{L/K}$ is given by Tate’s theorem when we take the fundamental class $u_{L/K}$ as our generator of $H^2(\text{Gal}(L/K), L^\times)$.

We will also view $\theta_{L/K} : K^\times \to \text{Gal}(L/K)^{ab}$ as a surjective map with kernel $N_{L/K}(L^\times)$.

**Lemma 4.2.** Let $K \subseteq E \subseteq L$ be a tower of finite Galois extensions of non-archimedean local fields. Then,

$$E^\times \xrightarrow{\theta_{L/E}} \text{Gal}(L/E)^{ab}$$

$$K^\times \xrightarrow{\theta_{L/K}} \text{Gal}(L/K)^{ab}$$

**Proof.** Let $G := \text{Gal}(L/K)$ and $H := \text{Gal}(L/E)$. Hence, $\text{Gal}(E/K) = G/H$. We have the following commutative diagram:

$$
\begin{array}{ccc}
E^\times & \xrightarrow{\theta_{L/E}} & \text{Gal}(L/E)^{ab} \\
\downarrow^{N_{E/K}} & & \downarrow \ \\
K^\times & \xrightarrow{\theta_{L/K}} & \text{Gal}(L/K)^{ab}
\end{array}
$$

$$
\begin{array}{ccc}
E^\times/N_{E/L}(E^\times) & = & \hat{H}^0(H, L^\times) \\
\downarrow^{N_{E/K}} & & \downarrow \ \\
K^\times/N_{L/K}(L^\times) & = & \hat{H}^0(G, L^\times)
\end{array}
$$

$$
\begin{array}{ccc}
\hat{H}^0(H, L^\times) & \xrightarrow{\Phi_{E/K}} & \hat{H}^2(H, \mathbb{Z}) \\
\downarrow^{\cong} & & \downarrow \ \\
\hat{H}^0(G, L^\times) & \xrightarrow{\Phi_{L/K}} & \hat{H}^2(G, \mathbb{Z}) \\
\downarrow^{\cong} & & \downarrow \ \\
H^{ab} & & G^{ab}
\end{array}
$$

$\square$
An alternative way to describe the local Artin map is via cup products (see [1]). For our purposes, we only need to look at the cup products on the 0-th and the 2-nd degree. For a Galois extension \( L/K \) of non-archimedean local fields, we have
\[
H^0(\text{Gal}(L/K), L^\times) \times H^2(\text{Gal}(L/K), \mathbb{Z}) \overset{\cup}{\to} H^2(\text{Gal}(L/K), L^\times)
\]
\((a, f) \mapsto a \cup f\)
such that
\[
(a \cup f)(g_1, g_2) = a \otimes_{\mathbb{Z}} f(g_1, g_2) = a^{f(g_1, g_2)}.
\]
To get the last equality, note that the \( \mathbb{Z} \)-action on \( K^\times \) is defined as follows: for \( z \in \mathbb{Z} \), \( a \in K^\times \),
\[
z \cdot a = a^z.
\]

**Lemma 4.3.** For every \( \chi \in \text{Hom}(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) = H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \) and \( a \in K^\times \),
\[
\chi(\theta_{L/K}(a)) = \text{inv}_{L/K}(a \cup \delta \chi)
\]
where \( \delta \) is the boundary map
\[
\delta : H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(\text{Gal}(L/K), \mathbb{Z}).
\]

In order to define the local Artin map from \( K^\times \) to \( \text{Gal}(K^{ab}/K) \), we need the following lemma:

**Lemma 4.4.** Suppose \( K \subseteq E \subseteq L \) is a tower of finite Galois extensions of non-archimedean local fields. Then,
\[
K^\times \xrightarrow{\theta_{L/K}} \text{Gal}(L/K) \\
\downarrow \quad \quad \quad \quad \quad \quad \theta_{E/K} \downarrow \quad \quad \quad \quad \quad \quad \text{Gal}(E/K)
\]
In other words,
\[
\theta_{L/K}(a)|_E = \theta_{E/K}(a)
\]
for all \( a \in K^\times \).

**Proof.** Notice that any homomorphism \( \chi : \text{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z} \) extends to a map \( \text{Gal}(L/K) \to \text{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z} \) which will still be called \( \chi \). Now to prove the lemma, it is enough to prove that
\[
\chi(\theta_{L/K}(a)) = \chi(\theta_{E/K}(a))
\]
for all \( \chi : \text{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z} \) and all \( a \in K^\times \). But then by using the formula in previous lemma, it suffices to prove that
\[
\text{inv}_{L/K}(a \cup \delta_{L/K} \chi) = \text{inv}_{E/K}(a \cup \delta_{E/K} \chi).
\]
But this follows from \( \text{inv}_{E/K} = \text{inv}_{L/K} \circ \text{Inf} \) and the fact that inflation map commutes with cup product. \( \square \)
Now we can extend the local Artin map to a maximal abelian extension \( K^{ab} \):

**Definition 4.5.** The local Artin map for a local field \( K \)

\[
\theta_K : K^\times \to \text{Gal}(K^{ab}/K)
\]

is a continuous group homomorphism defined by a compatible system of maps \( \{ \theta_{L/K} \} \).

The following theorem tells us the action of \( K^\times \) on the maximal unramified extension \( K^{ur} \) in \( K^{ab} \).

**Theorem 4.6.** Let \( \text{Frob}_K \) be the Frobenius element of \( K \) and \( v_K \) be the discrete valuation on \( K^\times \). Then,

\[
\theta_K(a)|_{K^{ur}} = \text{Frob}_K^{v_K(a)}
\]

for all \( a \in K^\times \). In particular, \( \theta_K(\pi) = \text{Frob}_K \) for any uniformizer \( \pi \) of \( K \).

**Proof.** By the definition of local Artin map, it is sufficient to show that

\[
\theta_{L/K}(a) = \text{Frob}_{L/K}^{v_L(a)}
\]

for every finite unramified extension \( L/K \).

Writing out the definition of the invariant map \( \text{inv}_{L/K} \), we have

\[
H^2(\text{Gal}(L/K), L^\times) \xrightarrow{v_L} H^2(\text{Gal}(L/K), \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{inv}_{L/K}} \mathbb{Q}/\mathbb{Z}.
\]

Let \( a \in K^\times \) and \( \chi : \text{Gal}(L/K) \to \mathbb{Q}/\mathbb{Z} \) be any homomorphism. Then under the maps,

\[
(a \cup \delta \chi) \mapsto v_L(a) \cup \delta \chi \mapsto v_L(a)\chi \mapsto \chi(\text{Frob}_{L/K}^{v_L(a)}) = \chi(\text{Frob}_{L/K}^{v_K(a)})
\]

and hence \( \text{inv}_{L/K}(a \cup \delta \chi) = \chi(\text{Frob}_{L/K}^{v_K(a)}) \). We can verify this in details: for the first arrow, given \( g_1, g_2 \in \text{Gal}(L/K) \),

\[
v_L((a \cup \delta \chi)(g_1, g_2)) = v_L((a \cup \delta \chi)(g_1, g_2)) = v_L(a \otimes \mathbb{Z} \delta \chi(g_1, g_2)) = v_L(a\delta \chi(g_1, g_2)) = \delta \chi(g_1, g_2)v_L(a)
\]

\[
(v_L(a) \cup \delta \chi)(g_1, g_2) = v_L(a) \otimes \mathbb{Z} \delta \chi(g_1, g_2) = v_L(a)\delta \chi(g_1, g_2)
\]

and so \( v_L(a \cup \delta \chi) = v_L(a) \cup \delta \chi \). For the second map, note that

\[
\delta(v_L(a)\chi) = \delta(v_L(a)) \cup \chi + (-1)^0v_L(a) \cup \delta \chi = v_L(a) \cup \delta \chi
\]

and so \( \delta^{-1}(v_L(a) \cup \delta \chi) = v_L(a)\chi \). The last map is just evaluation map.

On the other hand, by using the formula in lemma 4.3, we have

\[
\chi(\theta_{L/K}(a)) = \text{inv}_{L/K}(a \cup \delta \chi) = \chi(\text{Frob}_{L/K}^{v_K(a)})
\]

for all \( \chi \in H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \). It follows that

\[
\theta_{L/K}(a) = \text{Frob}_{L/K}^{v_K(a)}
\]

\( \Box \)
We conclude the section with a theorem that says there is no hope to extend the local Artin reciprocity theorems to non-abelian extensions.

**Theorem 4.7. (Norm Limitation Theorem)** Suppose \( L/K \) is a finite separable extension of non-archimedean local fields and \( E/K \) is the maximal abelian subextension in \( L \). Then,

\[
N_{L/K}(L^\times) = N_{E/K}(E^\times).
\]

**Proof.** By the transitivity of the norm map, \( N_{L/K} = N_{E/K} \circ N_{L/E} \), we see that \( N_{L/K}(L^\times) \subseteq N_{E/K}(E^\times) \).

We first assume \( L/K \) is Galois. Then the local Artin maps for \( L/K \) and \( E/K \) are given by

\[
\theta_{L/K} : K^\times / N_{L/K}(L^\times) \cong \text{Gal}(L/K)^{ab} = \text{Gal}(E/K)
\]
\[
\theta_{E/K} : K^\times / N_{E/K}(E^\times) \cong \text{Gal}(E/K)^{ab} = \text{Gal}(E/K).
\]

This shows that \( N_{L/K}(L^\times) = N_{E/K}(E^\times) \).

For a general case, let \( M/K \) be the Galois closure of \( L \). Denote \( G := \text{Gal}(M/K) \) and \( H := \text{Gal}(M/L) \). We claim that

\([G,G]H = \text{Gal}(M/E)\).

In fact we have the tower of Galois extensions:

\[
\begin{array}{cccccc}
& & M & & \\
& & \downarrow & & \\
L & \downarrow & M^{[G,G]} & \downarrow & E & \downarrow & K \\
& & \downarrow & & \\
& & E & & K
\end{array}
\]

By assumption, \( E \) is the maximal abelian extension in \( L \). On the other hand, \( M^{[G,G]} \) is the maximal abelian extension in \( M \), so we must have

\[ E = L \cap M^{[G,G]} = M^H \cap M^{[G,G]} = M^{[G,G]H} \]

and so by Galois correspondence \([G,G]H = \text{Gal}(M/E)\).

Consider the following diagram:

\[
\begin{array}{ccccccc}
L^\times & \xrightarrow{\theta_{M/L}} & H^{ab} & \xrightarrow{i} & H/[H,H] & \downarrow \\
N_{L/K} & \downarrow & \downarrow & \downarrow & \downarrow & \\
K^\times & \xrightarrow{\theta_{M/K}} & G^{ab} & \xrightarrow{\sigma} & G^{ab} & \downarrow \\
& & \downarrow & \downarrow & \downarrow & \\
& & \text{Gal}(E/K) & \xrightarrow{\sigma} & G/([G,G]H)
\end{array}
\]
We want to show that $N_{E/K}(E^\times) \subseteq N_{L/K}(L^\times)$. Take $a \in N_{E/K}(E^\times) = \ker(\theta_{E/K}) = \ker(\pi \circ \theta_{M/K})$. Then we have

$$\theta_{M/K}(a) \in \ker(\pi) = \text{Im}(i).$$

So, there exists $b \in L^\times$ such that

$$\theta_{M/K}(a) = \theta_{M/L}(b) = \theta_{M/K}(N_{L/K}(b)).$$

But this means

$$a \in N_{L/K}(b)\ker(\theta_{M/K}) = N_{L/K}(b)N_{M/K}(M^\times) = N_{L/K}(b) \cdot N_{L/K}(N_{M/L}(M^\times)) \subseteq N_{L/K}(L^\times)$$

as desired. \qed

References

[http://math.ucla.edu/~sharifi/lecnotes.html](http://math.ucla.edu/~sharifi/lecnotes.html)

[http://www.jmilne.org/math/CourseNotes/](http://www.jmilne.org/math/CourseNotes/)